On the Structure of Informationally Robust Optimal Auctions*

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Abstract

We study the design of profit-maximizing mechanisms in environments with interdependent values. A single unit of a good is for sale. There is a known joint distribution of the bidders' values for the good. Two programs are considered:

(i) Max (over mechanisms) min (over information structures and equilibria) profit;

(ii) Min (over information structures) max (over mechanisms and equilibria) profit.

We show that it is without loss to restrict attention to solutions of (i) and (ii) in which actions and signals belong to the same linearly ordered space, equilibrium actions are equal to signals, and the only binding equilibrium constraints are those associated with local deviations. Under such restrictions, the non-linear programs (i) and (ii) become linear programs that are dual to one another in an approximate sense. In particular, the restricted programs have the same optimal value, which we term the *profit guarantee*. These results simplify the task of computing and characterizing informationally robust optimal auctions and worst-case information structures with general value distributions. The framework can be generalized to include additional feasibility constraints, multiple goods, and ambiguous value distributions.

Keywords: Mechanism design, information design, optimal auctions, interdependent values, max-min, Bayes correlated equilibrium.

JEL Classification: C72, D44, D82, D83.

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1 Introduction

This paper presents new tools for optimal auction design when the seller knows the distribution of the bidders' values but does not know how to model the bidders' information. In particular, the seller is unable or unwilling to quantify their model uncertainty as a Bayesian prior over the bidders' higher-order beliefs—that is, an information structure—as required by the standard model of optimal auction design. We characterize mechanisms that guarantee the greatest level of profit uniformly across all common-prior information structures. Such informationally robust mechanisms are a natural starting point and benchmark in the face of model uncertainty about bidders' information.

We assume that while the information structure is unknown to the designer, the bidders' behavior in any given mechanism is described by an equilibrium under some information structure. This assumption is discussed in greater detail in the conclusion of this paper. But it immediately points to an important conceptual question: Can the seller not have the bidders simply report the information structure (which is common knowledge among them), and thereby directly condition the mechanism on the correct model of information? This seems especially plausible if we further assume that the seller chooses which equilibrium is played.[1] Of course, such mechanisms may be of limited practical value, since they require the bidders to articulate potentially complicated higher-order beliefs. A distinct issue is that the seller may be concerned about misspecification of both the information structure and the equilibrium, in which case there is no simple trick by which the mechanism can condition directly on the true information structure.

Our formal analysis concerns two optimization programs which describe a best guaranteed level of profit under different assumptions about whether the mechanism can condition directly on the information structure and how an equilibrium is selected. In the *min-2max* program, we minimize the seller's optimal profit across all information structures, where the seller can condition the mechanism on the information structure and select their preferred equilibrium. (The "2max" indicates that two objects are being chosen to maximize profit, the mechanism and the equilibrium.) In the *max-2min* program, we maximize the seller's worst-case profit across mechanisms, where the worst case is over information structures and equilibria. In particular, the mechanism cannot condition directly on the information structure and the seller's least-preferred equilibrium is selected.

Our main result is a characterization of the min-2max and max-2min programs that both reveals the structure of optimal solutions and simplifies their computation. In particular, we show that in both programs, it is without loss of generality to restrict attention to solutions in which actions and signals belong to the same linearly ordered space, equilibrium actions are equal to signals, and the only binding equilibrium conditions are those associated with local deviations. This pattern of binding constraints has further implications for min-2max information structures and max-2min mechanisms that we discuss below. Significantly, it implies that the inherently non-linear max-2min and min-2max programs can be restricted

[1] For example, if there are three or more bidders, the seller could use the following mechanisms: Each bidder reports the information structure. If a majority of the bidders report the same information structure, use that to design the mechanism, and otherwise design it for some arbitrary information structure. This game has an equilibrium where all bidders report the information structure truthfully.

in such a way that they become linear programs, without changing the respective values. Moreover, we show that restricted min-2max and max-2min programs actually have the same value, which we term the *profit guarantee*. A fortiori, this is also the value of the (unrestricted) max-2min and min-2max programs. Thus, the seller can achieve the same profit uniformly across information structures and equilibria, whether or not the mechanism can condition directly on the information structure and no matter how the equilibrium is selected. And it is possible to compute mechanisms and information structures that attain the profit guarantee by solving linear programs.

We now describe the restricted programs in greater detail, starting with min-2max. For a fixed information structure, the revelation principle for mechanism design implies that it is without loss in the inner "2max" program to restrict attention to direct revelation mechanisms—in which each action is a "reported" signal—and equilibria in which bidders report their true signals (Myerson, 1981). Holding information fixed, the problem of maximizing profit over truthful direct mechanisms is a linear program. Therefore, by the strong duality theorem of linear programming, one can view the min-2max program as a minimization problem, where we optimize over the information structure and the multipliers on participation and truthtelling constraints. This program is non-linear, because it involves the product of the probabilities in the information structure and the multipliers for the equilibrium constraints. We obtain a program with weakly higher value by fixing an (arbitrary) order on signals and restricting attention to information structures and multipliers for which only local incentive constraints bind.[2] We show that the binding local constraints imply that bidders' signals are independently distributed. The marginal on signals is indeterminate, and we simplify formulae by normalizing it to a censored geometric. The remaining choice variable is the correlation between signals and values. For information structures with independent signals and binding local constraints, it is well known that transfers can be solved out in terms of the allocation, and that expected profit is equal to the expected *virtual value* of the buyer that is allocated the good (Myerson, 1981; Bulow and Klemperer, 1996). A bidder's virtual value is equal to the expected value from being allocated the good, minus an *information rent* that is derived from the change in the bidder's surplus from a local deviation in the reported signal. Thus, an upper bound on the seller's expected profit is the expected maximum virtual value, where the expectation is across signal profiles and the maximization is across bidders and the seller (who has a virtual value of zero). Our restricted min-2max program simply minimizes this profit upper bound across information structures. This is a linear program, as once we have normalized the signal distribution, virtual values are linear in the joint distribution of signals and values.

The restricted max-2min program is based on a logic that is dual to that just described for min-2max. For a fixed mechanism, the revelation principle for information design says that it is without loss in the inner "2min" program to restrict attention to *Bayes correlated equilibria* (BCE), that is, information structures in which each bidder's signal is a recommended action, and in equilibrium, bidders obey their recommendations (Bergemann

[2]This restriction on solutions to the dual of the inner 2max program can also be viewed as a relaxation of the inner 2max program. The net effect is to restrict the feasible set for the min-2max program as a whole.

and Morris, 2013, 2016). Again, the problem of minimizing expected profit across BCE is a linear program, so we can view the max-2min program as a big (non-linear) maximization problem over mechanisms and multipliers on the constraints that characterize BCE. We obtain a program with weakly lower value by restricting attention to mechanisms and multipliers for which only local obedience constraints bind. We show that for BCE with binding local obedience, expected profit is equal to an expected *virtual profit*. This new object, which depends on the bidders' realized actions and values, is equal to the profit generated by the action profile, plus a *strategic rent*, which is the sensitivity of bidders' surpluses to local deviations in their actions. Thus, a lower bound on expected profit is the expected minimum virtual profit, where the expectation is over values and the minimum is over actions. Our restricted max-2min program simply maximizes this profit lower bound across all mechanisms. This is a linear program, since virtual profit is linear in the allocation and transfer rules.

In our formal analysis, we work with mechanisms with finitely many actions and information structures with finitely many signals, so that each "max" is really a "sup" and each "min" is really an "inf". For any bound on the number of actions and signals, the value of restricted min-2max is necessarily greater than that of restricted max-2min. We show that as the number of actions and signals goes to infinity, the gap between the programs disappears, and both values converge to the profit guarantee.[3] The proof of this convergence reveals a deep connection between the restricted programs. In fact, the programs are "almost dual" to one another, in that the dual of the min-2max restriction involves choosing a mechanism to maximize an expected lowest virtual profit. But there is an important difference between this dual and the restricted max-2min program: in the former, the strategic rent is derived from local deviations to lower actions, whereas in the latter, it is derived from local deviations to higher actions. Similarly, in restricted min-2max, information rents come from local downward deviations, whereas in the otherwise-identical dual to restricted max-2min, information rents come from local upward deviations. We show that optimal solutions to these programs are sufficiently regular that when the number of actions and signals is large, the optimal value of the restricted programs does not depend on the direction of local deviations.[4]

As noted above, binding local constraints in the restricted min-2max program imply that the marginal distribution of signals is independent. The corresponding phenomenon in the restricted max-2min program is that transfers can be partially solved out in terms of a simpler object, which is the difference between the divergence of the transfer rule and the sum of the transfers. We call this the *aggregate excess growth*. In the dual pairing of variables and constraints between the restricted programs, the aggregate excess growth is the multiplier on the constraint that the marginal distribution over signals matches its

[3]This result requires a technical assumption that the distribution of values has full support. The full-support assumption is not used in the derivation of the restricted programs, and in our examples that violate full-support, we show constructively that the programs have asymptotically the same value.

[4]The approximate duality between the restricted min-2max and max-2min programs is distinct from other applications of duality in optimal auction design, such as Vohra (2011), Daskalakis et al. (2017), and Cai et al. (2019). In each of these cases, the "primal" program is a standard Bayesian auction design problem with an exogenously given information structure. In contrast, in each of the programs in our approximate dual pair, both the information structure and the mechanism are endogenous.

normalized value. In practice, we have found the aggregate excess growth to be a convenient analytical tool. After proving our main result, we explore in detail the connection between the transfer rule and the aggregate excess growth, and how to go back and forth between the two. This is also illustrated in our examples, which we describe next.[5]

The new tools that we develop for informationally robust mechanism design can be applied in many settings. The restricted programs can be used to numerically compute approximate max-2min mechanisms and min-2max information structures. Based on simulations, one can then guess the functional form of solutions, and analytically verify that they attain the profit guarantee using virtual values and virtual profits. This research program was applied in Du (2018),[6] Bergemann et al. (2016), and Brooks and Du (2020, 2021).[7] The first three of these papers concerned the case of pure common values, i.e., the bidders all have the same value ex post. Brooks and Du (2020) in particular showed that the max-2min mechanisms have the form of "proportional auctions," in which the sum of bidders' allocations and the sum of bidders payments only depend on the sum of their actions, and individual allocations and payments are proportional to actions. In Brooks and Du (2021), we study a variation of the present model in which the seller knows only the expected value of each bidder and an upper bound on each bidder's value.[8] We showed that proportional auctions remain max-2min optimal when the bidders have the same expected value, and we generalize the proportional auction to the case in which bidders have different expected values.

A main contribution of this paper is to show, non-constructively, that the restricted programs can be formulated more generally, and that the solutions to these restricted programs are always approximate min-2max information structures and max-2min mechanisms. Thus, the guess-and-verify strategy pursued in earlier work is guaranteed to succeed for general value distributions, at least in an approximate sense. In Section 4, we present another application of our framework to the case where bidders can have different values ex post but the average value is common knowledge. This is in a sense the opposite of the common value case, in which there is common knowledge that the difference between the values is zero but there is uncertainty about the average value. We explicitly solve the max-2min and min-2max programs for known average values when there are two bidders and both bidders have the same ex ante expected value. The max-2min mechanism turns out to be essentially deterministic and fully allocates the good to whoever has a higher bid.

[5]This phenomenon is distinct from the manner in which interim transfers are solved out to obtain the formula for revenue in terms of virtual values in Myerson (1981), Bulow and Klemperer (1996), and the min-2max program. In particular, the profit guarantee in the max-2min program depends on the ex post transfer rule, whereas in the standard model, only interim transfers matter for incentives. Also, rather than solving out transfers completely, we reduce them to a simpler object, the aggregate excess growth.

[6]In Du (2018), virtual profit is used to bound expected profit from below, but the form of the auction was not motivated by simulations of the restricted max-2min program, in contrast to Bergemann et al. (2016) and Brooks and Du (2020, 2021).

[7]In the present model, we work with finite mechanisms and information structures, whereas these other papers worked with continuous action and signal spaces. Nonetheless, the logic underlying the verification step is the same. Note that Bergemann et al. (2016) and Brooks and Du (2020, 2021) simply report the guessed solution and not the motivating simulations from restricted min-2max and max-2min programs.

[8]The generalization to this case is described in Section 5.

We contribute to a growing literature on informationally robust auction design. Previous papers have studied this problem under the assumption that values are private, including Chung and Ely (2007), Yamashita (2016), Chen and Li (2018), and Che (2020). In contrast, our model allows for values to be interdependent. Yamashita and Zhu (2018) also study robust mechanism design with interdependent values, but they focus on conditions under which ex-post incentive compatible mechanisms are also max-min optimal when the seller-preferred equilibrium is selected. Other related studies of robust mechanism design include Neeman (2003), Brooks (2013), Yamashita (2015), Carroll (2017), Bergemann, Brooks, and Morris (2019), and the literature on algorithmic mechanism design (e.g., Hartline and Roughgarden, 2009).

The rest of this paper proceeds as follows. Section 2 describes our model. Section 3 presents our main results on the restricted programs. Section 4 presents an application when the average value is known. Section 5 presents several extensions and additional theoretical results. Section 6 concludes the paper with a discussion of our assumptions and directions for future research. An Appendix contains omitted proofs and an Online Appendix contains further results and numerical examples.

2 Model

One unit of a good is for sale to a finite group of bidders, indexed by $i = 1, \ldots, N$. Each bidder demands the unit at a value of $v_i \in \mathbb{R}_+$, with the joint distribution of values being given by $\mu \in \Delta(\mathbb{R}_+^N)$. We assume that μ has a finite support contained in $V = \prod_{i=1}^N V_i$, where $V_i \subseteq \mathbb{R}_+$ is a finite set of values for bidder i. We let $\overline{v} = \max_i \max V_i$.

An *information structure* consists of a finite set of signals S_i for each bidder, with $S = \prod_{i=1}^N S_i$, and a joint distribution $\sigma \in \Delta(V \times S)$ such that the marginal of σ on V is μ. An information structure is denoted by $\mathcal{I} = (S, \sigma)$. We denote by $\mathbf{I}(S)$ the set of information structures with signal space S. We denote by $\mathbf{I}$ the set of information structures.[9]

A *mechanism* consists of a finite set of actions A_i for each bidder, with $A = \prod_{i=1}^N A_i$; an allocation rule $q : A \to [0,1]^N$ and $\Sigma q(a) \leqslant 1$ for all $a \in A$;[10] and a transfer rule $t : A \to \mathbb{R}^N$. Thus, a mechanism is a triple $\mathcal{M} = (A, q, t)$. A mechanism is *participation secure* if for all i, there exists a "secure" action $a_i \in A_i$ such that $t_i(a_i, a_{-i}) = 0$ for all $a_{-i} \in A_{-i} = \prod_{j \neq i} A_j$. We let $\mathbf{M}(A)$ denote the set of participation-secure mechanisms with action space A. $\mathbf{M}$ is the set of finite mechanisms.

A mechanism and information structure $(\mathcal{M}, \mathcal{I})$ define a Bayesian game, in which bidder i's (behavioral) strategy is a mapping $b_i : S_i \to \Delta(A_i)$. A strategy profile $b = (b_1, \ldots, b_N)$ is identified with the kernel $b : S \to \Delta(A)$ where $b(s)$ is the product measure $\prod_{i=1}^N b_i(s_i)$. Expected profit from a strategy profile b of a game $(\mathcal{M}, \mathcal{I})$ is

$$\Pi(\mathcal{M}, \mathcal{I}, b) = \sum_{v \in V} \sum_{s \in S} \sum_{a \in A} \sum_{i=1}^N t_i(a) b(a|s) \sigma(s, v).$$

[9]The set of finite information structures exists because we can identify finite sets of signals with finite subsets of $\mathbb{N}$. Likewise for the set of finite mechanisms.

[10]Throughout the paper, we adopt the convention that for a vector $x \in \mathbb{R}^N$, Σx denotes the sum $x_1 + \cdots + x_N$.

Bidder i's preferences over strategy profiles are represented by the utility function

$$U_i(b) = \sum_{v \in V} \sum_{s \in S} \sum_{a \in A} (v_i q_i(a) - t_i(a)) b(a|s) \sigma(s, v).$$

A *(Bayes Nash) equilibrium* is a strategy profile b such that $U_i(b) \geqslant U_i(b'_i, b_{-i})$ for all i and strategies b'_i. We let $B(\mathcal{M}, \mathcal{I})$ denote the set of equilibria for the game $(\mathcal{M}, \mathcal{I})$, which is always non-empty since the mechanism and information structure are finite.

For a signal space S, we define

$$\Pi^{\text{MIN}-2\text{MAX}}(S) = \inf_{\mathcal{I} \in \mathbf{I}(S)} \sup_{\mathcal{M} \in \mathbf{M}} \sup_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b). \tag{1}$$

This is the smallest profit to which we can hold the seller, when we restrict attention to information structures for which the signal space is S and the seller-preferred equilibrium is played. Similarly, for an action space A, we define

$$\Pi^{\text{MAX}-2\text{MIN}}(A) = \sup_{\mathcal{M} \in \mathbf{M}(A)} \inf_{\mathcal{I} \in \mathbf{I}} \inf_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b). \tag{2}$$

This is the largest profit that the seller can guarantee, if the seller is restricted to mechanisms for which the action space is A, and the profit-minimizing equilibrium is played. Note that the values of these programs depend only on the cardinality of the action and signal space. Our primary objective in this paper is to characterize solutions to these programs.

3 Results

We now present our main results. First we define the linear programs described in the introduction that are restrictions of (1) and (2). We then state our main result, with the proof immediately following.

3.1 Restricted programs

For each $k \in \mathbb{N}$, let

$$X_i(k) = \left\{ \frac{l}{k} \,\middle|\, 0 \leqslant l \leqslant k^2, l \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ is the integers, and $X(k) = \prod_{i=1}^{N} X_i(k)$. Given a function $f : X(k) \to \mathbb{R}^N$, the discrete upward partial derivative $\nabla_i^+ f(x)$ is[11]

$$\nabla_i^+ f(x) = \mathbb{I}_{x_i < k}(k-1)(f_i(x_i + 1/k, x_{-i}) - f_i(x)).$$

[11] Given that the increment between elements in $X(k)$ is $1/k$, a seemingly more natural definition of the discrete derivative would have a factor k rather than $k-1$. Of course, these definitions are equivalent in the limit as k tends to infinity, and by defining it with $k-1$, we simplify several calculations in the proof of Theorem 1.

We let $\nabla^+ f(x) = (\nabla_1^+ f(x), \ldots, \nabla_N^+ f(x))$. The discrete upward divergence is $\nabla^+ \cdot f(x) = \sum_{i=1}^N \nabla_i^+ f(x)$. Also, let

$$\rho_i(x_i) = \left(1 - \frac{1}{k}\right)^{kx_i} \frac{1}{k^{\mathbb{I}_{x_i<k}}}$$

denote the censored geometric distribution on $X_i(k)$ with arrival rate $1/k$, and

$$\rho(x) = \prod_{i=1}^N \rho_i(x_i)$$

be the product distribution on $X(k)$.

We now define the restrictions of (1) and (2) that we described in the introduction. The restriction of (1) (with $S = X(k)$) is

$$\begin{aligned}
\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) = \quad & \min_{\sigma:X(k)\times V\to\mathbb{R}_+,\ w:X(k)\to\mathbb{R}_+^N,\ \gamma:X(k)\to\mathbb{R}_+} \sum_{x\in X(k)} \gamma(x) \\
\text{s.t.} \quad & \sum_{x\in X(k)} \sigma(x,v) = \mu(v)\ \forall v; \\
& \sum_{v\in V} \sigma(x,v) = \rho(x)\ \forall x; \\
& w(x) = \frac{1}{\rho(x)} \sum_{v\in V} v\sigma(x,v)\ \forall x; \\
& \gamma(x) \geqslant \rho(x)\left[w_i(x) - \nabla_i^+ w(x)\right]\ \forall x, i.
\end{aligned} \tag{3}$$

This program involves choosing an information structure with signal space $X(k)$, and the first constraint is simply part of the definition of an information structure. But we have imposed an additional constraint, which is that the marginal distribution of the signals is ρ.[12] Given this fixed marginal on signals, the third constraint defines $w(x)$ to be the bidders' interim expected values, conditional on the signal profile x. The last constraint simply says that $\gamma(x)/\rho(x)$ must be at least the *virtual value* of bidder i. This is the social value from allocating to bidder i, which is $w_i(x)$, minus an information rent that bidder i accrues when allocated the good at the signal profile x, which is $\nabla_i^+ w(x)$. This expression is the analogue of the classic Myersonian formula, adapted to the case of discrete signals and interdependent values, and where we have assumed that each bidder's signal is a censored geometric random variable.[13] Thus, the program (3) corresponds to choosing

[12] As we discuss in the next section, the choice of ρ is merely a normalization; the substantive assumption is that signals are independent and that the limit signal distribution when $k \to \infty$ is absolutely continuous.

[13] The generalized formula for the virtual value from Bulow and Klemperer (1996) is

$$w_i(x) - h(x_i)\frac{\partial w_i(x_i)}{\partial x_i},$$

where $h_i(x_i)$ is the inverse hazard rate of bidder i's signal evaluated at x_i. In the case of private values, $w_i(x) = x_i$, and the formula reduces to that reported in Myerson (1981). For the censored geometric distribution, we have $h_i(x_i) = 1$.

the information structure to minimize the expected maximum virtual value, where the expectation is across signal profiles and the maximum is across bidders and the seller (where the seller's virtual value is implicitly taken to be zero).

The restriction of (2) (with $A = X(k)$) is

$$\begin{aligned} \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) = & \max_{q:X(k)\to\mathbb{R}^N_+,\ t:X(k)\to\mathbb{R}^N,\ \lambda:V\to\mathbb{R}} \sum_{v\in V} \mu(v)\lambda(v) \\ & \text{s.t. } \Sigma q(x) \leqslant 1\ \forall x; \\ & \quad t_i(0, x_{-i}) = 0\ \forall i, x_{-i}; \\ & \quad \lambda(v) \leqslant \Sigma t(x) + v \cdot \nabla^+ q(x) - \nabla^+ \cdot t(x)\ \forall v, x. \end{aligned} \tag{4}$$

The choice variables (q, t) are simply a participation secure mechanism with action space $X(k)$. Feasibility of the allocation and participation security are the first two constraints (where we have taken 0 to be the secure action). The third and most substantive constraint essentially says that $\lambda(v)$ is a lower bound on the seller's surplus when the value profile is v, and where the lower bound involves both revenue and bidders' local incentives. The right-hand side of this constraint is the *virtual profit* that we referred to in the introduction, and the term $v \cdot \nabla^+ q - \nabla^+ \cdot t$ is what we termed the *strategic rent*. The objective is to maximize the expected minimum virtual profit, where the expectation is across values and the minimum is across action profiles.

3.2 Main result

Our main result is the following:

Theorem 1. *For all k,*

$$\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \geqslant \Pi^{\text{MIN}-2\text{MAX}}(X(k)) \geqslant \Pi^{\text{MAX}-2\text{MIN}}(X(k)) \geqslant \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k). \tag{5}$$

Any optimal solution of problem (3) *yields an information structure such that the maximum profit across mechanisms and equilibria is at most $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$, and any optimal solution of problem* (4) *yields a mechanism such that the minimum profit across information structure and equilibria is at least $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$. If $\mu(v) > 0$ for all $v \in V$, then there exists a Π^* such that*

$$\lim_{k\to\infty} \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) = \lim_{k\to\infty} \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) = \Pi^*. \tag{6}$$

Thus, the linear programs (3) and (4) bound (1) and (2), and under the full-support hypothesis, their values are asymptotically equal to the *profit guarantee* Π^*, in the limit as k goes to infinity. Theorem 1 does not hold when we impose an upper bound on the number of actions/signals. In simulations reported below, the value Π^* is only attained in the limit.

3.3 Proof of Theorem 1

We now prove Theorem 1. The first step is showing the chain of inequalities (5). We then show that $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$ converges to zero under the full support hypothesis.

3.3.1 Ordering $\Pi^{\text{MIN}-2\text{MAX}}$ and $\Pi^{\text{MAX}-2\text{MIN}}$

An elementary observation is that $\Pi^{\text{MIN}-2\text{MAX}}$ is always greater than $\Pi^{\text{MAX}-2\text{MIN}}$, and these values move closer as the the number of actions and signals increases.

Lemma 1. *For all S and A, $\Pi^{\text{MIN}-2\text{MAX}}(S) \geqslant \Pi^{\text{MAX}-2\text{MIN}}(A)$. Moreover, if $|A_i| \leqslant |A'_i|$ (respectively $|S_i| \leqslant |S'_i|$) for all i, then $\Pi^{\text{MAX}-2\text{MIN}}(A) \leqslant \Pi^{\text{MAX}-2\text{MIN}}(A')$ (respectively $\Pi^{\text{MIN}-2\text{MAX}}(S) \geqslant \Pi^{\text{MIN}-2\text{MAX}}(S')$).*

The first part follows from a standard argument in zero-sum games: In particular, when the seller moves second (as in (1)), they can always pick the mechanism that is approximately optimal for (2) with the action space A and an arbitrary equilibrium, which must give a value that is approximately at least $\Pi^{\text{MAX}-2\text{MIN}}(A)$. The second part follows from the fact that any mechanism with fewer actions can be replicated when there are more actions by treating some of the actions as copies, and any information structure with fewer signals can be replicated when there are more signals by assigning zero probability to the extra signals. See the Appendix for the complete proof.

3.3.2 Local restrictions

We now prove a key step in the proof of Theorem 1, which is that the programs (3) and (4) are indeed restrictions of (1) and (2), respectively, when the signal and action spaces are taken to be $X(k)$.

Lemma 2. *For all $k \in \mathbb{N}$, $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \geqslant \Pi^{\text{MIN}-2\text{MAX}}(X(k))$. Moreover, if $(\gamma^*, \sigma^*, w^*)$ is an optimal solution to (3), then $(X(k), \sigma^*)$ is an information structure for which profit in any mechanisms and equilibrium is at most $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$.*

Proof of Lemma 2. Consider the inner maximization program in (1) for a fixed information structure $\mathcal{I} = (X(k), \sigma)$ in $\mathbf{I}(X(k))$, in which we maximize expected profit over all participation-secure mechanisms and equilibria. The presence of the participation-security action implies that all bidders must receive non-negative utility in equilibrium. Thus, we can relax the program by dropping the requirement of participation security, and replacing it with the constraint that equilibrium interim bidder surpluses must be non-negative.

By the revelation principle (Myerson, 1981), this relaxed program is equivalent to maximizing expected profit over incentive compatible and individually rational direct mechanisms. Recall that a *direct mechanism* on the information structure $\mathcal{I}$ is a mechanism with $A = X(k)$. When the action and signal spaces coincide, we let $\overline{b}_i$ denote the *truthful strategies* that place probability one on $a_i = s_i$ for all i. The direct mechanism is *incentive compatible* if $\overline{b}$ is an equilibrium. It is *individually rational* if the truthful strategies give each bidder a non-negative expected payoff conditional on their signal. Thus, the relaxed

program is the linear program

$$\begin{aligned}
&\max_{q:S(k)\to\mathbb{R}^N_+,\, t:X(k)\to\mathbb{R}^N} \sum_{x\in X(k)}\sum_{v\in V} \Sigma t(x)\sigma(x,v)\\
\text{s.t. } &\sum_{v\in V}\sum_{x_{-i}\in X_{-i}(k)} (v_i q_i(x_i,x_{-i}) - t_i(x_i,x_{-i}))\sigma(x_i,x_{-i},v)\\
&\qquad \geqslant \sum_{v\in V}\sum_{x_{-i}\in X_{-i}(k)} (v_i q_i(x_i',x_{-i}) - t_i(x_i',x_{-i}))\sigma(x_i,x_{-i},v)\ \forall i, x_i, x_i';\\
&\sum_{v\in V}\sum_{x_{-i}\in X_{-i}(k)} (v_i q_i(x_i,x_{-i}) - t_i(x_i,x_{-i}))\sigma(x_i,x_{-i},v) \geqslant 0\ \forall i, x_i;\\
&\Sigma q(x) \leqslant 1\ \forall x.
\end{aligned}$$

By the strong duality theorem, the value of this program is equal to that of its dual:

$$\min_{\{\alpha_i:X_i(k)^2\to\mathbb{R}_+,\, \beta_i:X_i(k)\to\mathbb{R}_+\}_{i=1}^N,\, \gamma:X(k)\to\mathbb{R}_+} \sum_{x\in X(k)} \gamma(x) \tag{7}$$

$$\begin{aligned}
\text{s.t. } \gamma(x) \geqslant &\sum_{x_i'\in X_i(k)}\sum_{v\in V} v_i[\sigma(x,v)\alpha_i(x_i,x_i') - \sigma(x_i',x_{-i},v)\alpha_i(x_i',x_i)]\\
&+ \sum_{v\in V}\beta_i(x_i)v_i\sigma(x,v)\ \forall i, x;
\end{aligned} \tag{7a}$$

$$\begin{aligned}
\sum_{v\in V}\sigma(x,v) = &\sum_{x_i'\in X_i(k)}\sum_{v\in V}[\sigma(x,v)\alpha_i(x_i,x_i') - \sigma(x_i',x_{-i},v)\alpha_i(x_i',x_i)]\\
&+ \sum_{v\in V}\beta_i(x_i)\sigma(x,v)\ \forall i, x,
\end{aligned} \tag{7b}$$

where $\alpha_i(x_i, x_i')$ is the multiplier on the incentive compatibility constraint for type x_i not misreporting as x_i' and $\beta_i(x_i)$ is the multiplier on the individual rationality constraint for type x_i. Thus, (1) has a value less than or equal to the (non-linear) program of minimizing the value of (7) across all information structures σ and multipliers (α, β, γ).

Note that the value of the inner program (7) will only increase if we hold α and β fixed at particular values. In particular, consider the following feasible solution:

$$\alpha_i(x_i,x_i') = \begin{cases} 1 & \text{if } x_i' + \frac{1}{k} = x_i = k;\\ k & \text{if } x_i' + \frac{1}{k} = x_i < k;\\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and

$$\beta_i(x_i) = \begin{cases} k & \text{if } x_i = 0;\\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The constraint (7b) (for which $t_i(x)$ is the multiplier) can be simplified as follows: In a slight abuse of notation, let $\sigma(x)$ denote the marginal of σ on $X(k)$. Then integrating out

values and using the particular multipliers, (7b) becomes

$$\sigma(x) = \begin{cases} \frac{k-1}{k}\sigma(x_i - 1/k, x_{-i}) & \text{if } 0 < x_i < k; \\ (k-1)\sigma(k - 1/k, x_{-i}) & \text{if } x_i = k. \end{cases}$$

For each i and x_{-i}, this is a first-order difference equation in x_i, which has a unique solution given the boundary value $\sigma(0,0)$. The unique solution that is also a probability distribution is $\sigma(x) = \rho(x)$. As a result, we can replace (7b) with the constraint

$$\sum_{v \in V} \sigma(x, v) = \rho(x). \tag{10}$$

Thus, the multipliers in (8) and (9) are feasible for problem (7) if and only if the marginal distribution of σ on $X(k)$ is ρ.

In addition, substituting the chosen multipliers into (7a), the constraint becomes

$$\gamma(x) \geqslant \begin{cases} k \sum_{v \in V} v_i[\sigma(x, v) - \sigma(x_i + 1/k, x_{-i}, v)] & \text{if } x_i < k - 1/k; \\ \sum_{v \in V} v_i[k\sigma(x, v) - \sigma(x_i + 1/k, x_{-i}, v)] & \text{if } x_i = k - 1/k; \\ \sum_{v \in V} v_i \sigma(x, v) & \text{if } x_i = k. \end{cases} \tag{11}$$

Letting

$$w(x) = \frac{1}{\rho(x)} \sum_{v \in V} v\sigma(x, v) \tag{12}$$

denote the interim expected value of bidder i conditional on the signal profile x, the constraint (11) can be rewritten as

$$\gamma(x) \geqslant \rho(x) \left[w_i(x) - \nabla_i^+ w(x) \right]. \tag{13}$$

Thus, replacing (7b) with (10) and replacing (7a) with (12) and (13) yields a program with weakly higher value than (1). This program is precisely (3). □

Lemma 3. *For all $k \geqslant 0$, $\underline{\Pi}^{\text{MAX-2MIN}}(k) \leqslant \Pi^{\text{MAX-2MIN}}(X(k))$. Moreover, if (λ^*, q^*, t^*) is an optimal solution of problem (4), then $(X(k), q^*, t^*)$ is a mechanism for which profit in any information structure and equilibrium is at least $\underline{\Pi}^{\text{MAX-2MIN}}(k)$.*

Proof of Lemma 3. Consider the inner minimization program in (2) for a fixed mechanism $\mathcal{M} = (X(k), q, t)$ in $\mathbf{M}(X(k))$. The program of minimizing expected profit over all information structures and equilibria can be reformulated as a linear program. Specifically, a *Bayes correlated equilibrium* (BCE) of $\mathcal{M}$ is an information structure with $S = X(k)$ such that the obedient strategies $\bar{b}$ are an equilibrium. The problem of minimizing expected profit over information structures and equilibria is equivalent to minimizing expected profit over

BCE (Bergemann and Morris, 2013, 2016). Explicitly, this program is

$$\begin{aligned}
&\min_{\sigma:X(k)\times V\to\mathbb{R}_+} \sum_{v\in V}\sum_{x\in X(k)} \Sigma t(x)\sigma(x,v)\\
\text{s.t.}\ &\sum_{v\in V}\sum_{x_{-i}\in X_{-i}(k)} \left[v_i q_i(x_i,x_{-i}) - t_i(x_i,x_{-i})\right]\sigma(x_i,x_{-i},v)\\
&\qquad \geqslant \sum_{v\in V}\sum_{x_{-i}\in X_{-i}(k)} \left[v_i q_i(x_i',x_{-i}) - t_i(x_i',x_{-i})\right]\sigma(x_i,x_{-i},v)\ \forall i,x_i,x_i';\\
&\sum_{x\in X(k)} \sigma(x,v) = \mu(v)\ \forall v.
\end{aligned}$$

The value of this program is equal to that of its dual:

$$\begin{aligned}
&\max_{\{\alpha_i:(X_i(k))^2\to\mathbb{R}_+\}_{i=1}^N,\,\lambda:V\to\mathbb{R}} \sum_{v\in V}\lambda(v)\mu(v) \qquad (14)\\
\text{s.t.}\ &\lambda(v) \leqslant \Sigma t(x) + \sum_{i=1}^N\sum_{x_i'\in X_i(k)} \alpha_i(x_i,x_i')\big[(v_i q_i(x_i',x_{-i}) - t_i(x_i',x_{-i}))\\
&\qquad\qquad - (v_i q_i(x_i,x_{-i}) - t_i(x_i,x_{-i}))\big]\ \forall x,v,
\end{aligned}$$

where $\alpha(x_i,x_i')$ is the multiplier on the obedience constraint that a bidder with signal x_i not want to bid x_i', and $\lambda(v)$ is the multiplier on the constraint that the marginal probability of $v\in V$ under σ is $\mu(v)$. Moreover, any feasible solution to the dual is a lower bound on the value of the primal. In particular, consider the following feasible multipliers:

$$\alpha_i(x_i,x_i') = \begin{cases} k-1 & \text{if } x_i' - \frac{1}{k} = x_i;\\ 0 & \text{otherwise.}\end{cases}$$

In this case, the dual constraint becomes

$$\lambda(v) \leqslant \Sigma t(x) + v\cdot\nabla^+ q(x) - \nabla^+\cdot t(x). \qquad (15)$$

Thus, the maximum of (14) subject to (15) is a lower bound on the inner minimization program in (2). As a result, the maximum of this lower bound across all participation secure mechanisms, given by the linear program (4), is a lower bound on the value of (2). □

To summarize, the restricted programs (3) and (4) are derived from (1) and (2) by fixing an (arbitrary) order on the signal/action space, and dropping all incentive/obedience constraints except those that are associated with local deviations, and fixing a particular set of multipliers on local constraints (which corresponds to an implicit restriction to primal solutions for which local constraints bind). Obviously, the particular labeling of the signals and actions is arbitrary, and we have simply normalized those labels so that the order in which local constraints bind coincides with the usual order on $\mathbb{R}$.

With regard to the program (3), the proof of Lemma 2 shows that, rather remarkably, binding local incentive compatibility actually implies that bidders' signals are independent.

The fact that the signal distribution is exactly ρ is an artifact of the particular choice of multiplier on local obedience.[14] Lemma 2 also implicitly uses binding local incentive compatibility to solve out the interim transfer and rewrite expected profit as the expected virtual value of the winner (Myerson, 1981). This virtual value implicitly depends on the independent distribution of signals, which is also ρ.

3.3.3 Convergence

We now sketch the last part of the proof of Theorem 1, which shows that the programs (3) and (4) have the same value in the limit as k goes to infinity. In this limit, the programs (3) and (4) are "close" to being a dual pair. To see this, we first observe that (3) has the following dual:[15]

$$\begin{aligned} &\max_{\lambda:V\to\mathbb{R},\,\Xi:X(k)\to\mathbb{R},\,q:X(k)\to\mathbb{R}^N_+} \sum_{x\in X(k)} \rho(x)\Xi(x) + \sum_{v\in V}\mu(v)\lambda(v) \\ \text{s.t. } &\Xi(x) + \lambda(v) \leqslant v\cdot\nabla^- q(x)\ \forall v, x; \\ &\Sigma q(x) \leqslant 1\ \forall x, \end{aligned} \tag{3-D}$$

where

$$\nabla_i^- q(x) = \begin{cases} kq_i(x) & x_i = 0; \\ k(q_i(x) - q_i(x_i - 1/k, x_{-i})) & 0 < x_i < k; \\ q_i(x) - q_i(x_i - 1/k, x_{-i}) & x_i = k \end{cases}$$

is the discrete downward derivative of q.

This program is similar to (4), except for two key differences: First, (3-D) has a new choice variable $\Xi(x)$, which is substituted in the constraint in place of the term $\nabla\cdot t(x) - \Sigma t(x)$. We refer to the latter as the *aggregate excess growth*. There is also a new term in the objective, the expectation of Ξ under ρ. In fact, we can make a similar substitution of Ξ for t in the program (4), as we now explain. Let us say that $\Xi : X(k) \to \mathbb{R}$ is *balanced* if

$$\sum_{x\in X(k)} \rho(x)\Xi(x) = 0. \tag{16}$$

[14] With a general multiplier on local obedience, the constraint (7b) reduces to

$$\sigma_i(x_i) = \sigma_i(x_i)\alpha_i(x_i, x_i - 1/k)\mathbb{I}_{x_i>0} - \sigma(x_i + 1/k)\alpha_i(x_i + 1/k, x_i)\mathbb{I}_{x_i<k}\ \forall i, x,$$

where we have integrated out v and x_{-i} and let $\sigma_i(x_i)$ denote the marginal distribution of bidder i's signal. Summing the above equation across $x_i' \geqslant x_i$ yields

$$\sum_{x_i'\geqslant x_i} \sigma(x_i') = \sigma(x_i)\alpha_i(x_i, x_i - 1/k).$$

In other words, the multiplier on the local incentive compatibility constraint is exactly equal to the inverse hazard rate of the signal distribution.

[15] When taking the dual of (3) we find it convenient to replace the constraint (13) by the equivalent constraint (11).

We have the following lemma[16]

Lemma 4. *Given $\Xi : X(k) \to \mathbb{R}$, there exists a t that solves*

$$\nabla^+ \cdot t(x) - \Sigma t(x) = \Xi(x) \ \forall x; \tag{17}$$

$$t_i(0, x_{-i}) = 0 \ \forall i, x_{-i} \tag{18}$$

if and only if Ξ is balanced.

Proof of Lemma 4. By Fredholm's alternative, there exists a t that solves (17) and (18) if and only if there does not exist a ρ' such that

$$\sum_{x \in X(k)} \rho'(x)\Xi(x) \neq 0 \tag{19}$$

$$\rho'(x) = \begin{cases} \frac{k-1}{k}\rho'(x_i - 1/k, x_{-i}) & \text{if } 0 < x_i < k; \\ (k-1)\rho'(k - 1/k, x_{-i}) & \text{if } x_i = k. \end{cases}$$

It is easy to see that the choice of $\rho'(0)$ pins down the rest of ρ', and in fact

$$\rho'(x) = \rho(x)\frac{\rho'(0)}{\rho(0)}.$$

As a result, (19) holds if and only if $\sum_{x \in X(k)} \rho(x)\Xi(x) \neq 0$. Thus, (17) and (18) has a solution if and only if Ξ is balanced. □

In light of Lemma 4, the program (4) would remain the same if we substitute $\Xi(x)$ for $\nabla^+ \cdot t(x) - \Sigma t(x)$, and added the constraint that Ξ is balanced. But clearly, the balance condition (16) is just a normalization: If the expectation of Ξ under ρ is $C \neq 0$, then we could define a new solution where $\lambda' = \lambda + C$ and $\Xi' = \Xi - C$, so that Ξ' is balanced and the objective is the same. Thus, the value of the program would be the same if, instead of imposing the constraint (16), we simply add the left-hand side of (16) to the objective, to obtain the following program:

$$\max_{\lambda: V \to \mathbb{R},\, \Xi: X(k) \to \mathbb{R},\, q: X(k) \to \mathbb{R}^N_+} \sum_{x \in X(k)} \rho(x)\Xi(x) + \sum_{v \in V} \mu(v)\lambda(v) \tag{4'}$$

$$\text{s.t. } \Xi(x) + \lambda(v) \leqslant v \cdot \nabla^+ q(x) \ \forall v, x; \tag{4'a}$$

$$\Sigma q(x) \leqslant 1 \ \forall x, \tag{4'b}$$

By the argument in the preceding paragraph, we have the following lemma:

Lemma 5. *The programs* (4) *and* (4′) *have the same value of* $\underline{\Pi}^{\text{MAX-2MIN}}(k)$.

[16] The aggregate excess growth played a prominent role in our earlier analysis of common values in Brooks and Du (2020). In that paper, we gave a constructive argument for the existence of participation secure transfers when Ξ is balanced. We comment further on this connection in Section 3.4.

The remaining key difference between the program (3-D) (which is the dual of (3)) and (4′) (which is a reformulation of (4)) is that the term $\nabla^+ q$ in the former is replaced with $\nabla^- q$ in the latter. Thus, incentive constraints point in the opposite direction in the two programs. Intuitively, this difference should be immaterial when k is large, as long as the solutions to (3-D) are sufficiently smooth. This is the sense in which the programs (3) and (4) are "almost" a dual pair.[17] This is indeed the case:

Lemma 6. *Suppose $\mu(v) > 0$ for all v. For any $\epsilon > 0$, there exists a K such that for all $k \geqslant K$, $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \geqslant \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \epsilon$.*

The proof of Lemma 6 in Appendix A shows that when k is large, for any q that is optimal for (3-D), we can find a q' that is feasible for (4′) such that $\nabla^- q$ and $\nabla^+ q'$ are approximately equal. This step takes considerable effort to prove. The high-level strategy is as follows: Given a q that is feasible for (3-D), a natural way to define q' would be to simply "shift" the allocation up to the next higher signal, i.e., set

$$q_i'(x) = q_i(x_i - 1/k, x_{-i})$$

when $x_i > 0$, and set $q_i(0, x_{-i}) = 0$, so that $\nabla^+ q'(x) = \nabla^- q(x)$ for all x. The problem is that q' defined in this manner might be infeasible because $\Sigma q'(x) > 1$. This could happen if q is decreasing at x. Now, as long as the amount by which q decreased was small, we could simply normalize by replacing q' with $q'/\max\{\Sigma q', 1\}$ without significantly changing the objective. Lemma 10 shows that there exists a constant C such that for all k, if q is optimal for (3-D), then

$$q_i(x_i - 1/k, x_{-i}) \leqslant q_i(x) + \frac{C}{k}. \tag{20}$$

Thus, the aforementioned normalization will have only a small effect on the allocation when k is large. It is in establishing (20) where we use the full support hypothesis, that $\mu(v) > 0$ for all $v \in V$.[18]

With Lemma 6 in hand, we now complete the proof of Theorem 1.

[17] Note that the dual of (4′) (which is a reformulation of the dual of (4)) is similarly "almost" equivalent to (3). In particular, both involve minimization over information structures whose marginal on $X(k)$ is ρ, and $\gamma(x)$ (the multiplier on feasibility of the allocation) represents a highest "virtual value" of the bidders. The key difference is that in (3), the virtual value, written in terms of the interim value, is

$$w_i(x) - (k-1)(w_i(x_i + 1/k, x_{-i}) - w_i(x))$$

whereas in the dual of (4), the virtual value is

$$w_i(x) - k(w_i(x) - w_i(x_i - /k, x_{-i})).$$

Thus, the key difference between these expressions is whether local upward or local downward constraints are used to compute information rents, which is immaterial when k is large and as long as the optimal value function is sufficiently smooth.

[18] In fact, the full support hypothesis is only used to establish an even weaker result, that optimal λ are bounded as $k \to \infty$.

Proof of Theorem 1. The ranking (5) follows immediately from Lemmas 1, 2, and 3. That optimal solutions to (3) and (4) guarantee profit of $\Pi^{\text{MIN}-2\text{MAX}}(k)$ and $\Pi^{\text{MAX}-2\text{MIN}}(k)$, respectively, follows from Lemmas 2 and 3.

Now, Lemma 1 implies that $\lim_{k\to\infty} \Pi^{\text{MIN}-2\text{MAX}}(X(k)) = \overline{\Pi}$ and $\lim_{k\to\infty} \Pi^{\text{MAX}-2\text{MIN}}(X(k)) = \underline{\Pi}$ both exist, and moreover, for all k,

$$\Pi^{\text{MIN}-2\text{MAX}}(X(k)) \geqslant \overline{\Pi} \geqslant \underline{\Pi} \geqslant \Pi^{\text{MAX}-2\text{MIN}}(X(k)).$$

Combining Lemma 6 with (5), we have that for any $\epsilon > 0$, there exists a K such that for all $k \geqslant K$,

$$\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \geqslant \overline{\Pi} \geqslant \underline{\Pi} \geqslant \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \geqslant \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \epsilon.$$

Hence, $\overline{\Pi} - \underline{\Pi} \leqslant \epsilon$. Since ϵ was arbitrary, we conclude that $\overline{\Pi} = \underline{\Pi} = \Pi^*$. This implies that

$$\Pi^* \geqslant \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \geqslant \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \epsilon \geqslant \Pi^* - \epsilon$$

for all $k \geqslant K$. Since $\epsilon > 0$ was arbitrary, we conclude that $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$ and $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$ both converge to Π^*, as desired. □

3.4 Transfers

We conclude this section with an analysis of max-2min transfers, that is, transfers that are part of an optimal solution to (4), and how they are related to the max-2min allocation. The proof of Theorem 1, and Lemma 4 in particular, shows that we can essentially solve out the transfers from (4) in terms of the aggregate excess growth Ξ, to obtain the equivalent program (4′). In applications, we have found that it is often more convenient to work with the program (4′), and derive the optimal allocation q and multipliers (λ, Ξ).

Importantly, Lemma 4 tells us that it is possible to go back and forth between solutions of the programs (4) and (4′). In particular, given (λ, q, t) that is feasible for (4), we can define Ξ according to (17). Lemma 4 implies that Ξ is balanced, so that (λ, Ξ, q) is feasible for (4′) and has the same value as (λ, q, t) in (4). In the other direction, given (λ, Ξ, q) that is feasible for (4′), there is another optimal solution $(\lambda + C, \Xi - C, q)$ such that $\Xi - C$ is balanced. Lemma 4 then implies that there exists a transfer rule t with aggregate excess growth $\Xi - C$ and satisfies participation security (18), so that $(\lambda + C, q, t)$ is feasible for (4) and has the same value. This discussion is formalized in the following corollary:

Corollary 1. *The triple (λ^*, q^*, t^*) is an optimal solution to (4) only if (λ^*, Ξ^*, q^*) is an optimal solution to (4′), where $\Xi^* = \nabla^+ \cdot t^* - \Sigma t^*$. The triple (λ^*, Ξ^*, q^*) is an optimal solution to (4′) only if there is a $C \in \mathbb{R}$ and a t^* where $\Xi^* - C = \nabla^+ \cdot t^* - \Sigma t^*$ and $(\lambda^* + C, q^*, t^*)$ is an optimal solution to (4).*

The argument in Lemma 4 is non-constructive. But in fact, given a balanced Ξ, it is straightforward to construct a transfer rule t that is participation secure and has the given aggregate excess growth. To develop this result, let us first denote by $\xi_i(x)$ bidder i's *individual excess growth*:

$$\xi_i(x) \equiv \nabla_i^+ t(x) - t_i(x). \tag{21}$$

We can view this as a first-order difference equation in x_i, which we can use to solve for the transfer in terms of the individual excess growth. The solution that satisfies the boundary condition $t_i(k, x_{-i}) = -\xi_i(k, x_{-i})$ (which is just equation (21) when $x_i = k$) is

$$t_i(x) = -\sum_{y_i : y_i \geqslant x_i} \left(\frac{k-1}{k}\right)^{(y_i - x_i)k} \frac{1}{k^{\mathbb{I}_{y_i < k}}} \xi_i(y_i, x_{-i}).$$

Using the definition of ρ_i, we can rewrite this more simply as

$$t_i(x) = -\frac{1}{\rho_i(x_i)} \sum_{y_i : y_i \geqslant x_i} \rho_i(y_i) \xi_i(y_i, x_{-i}). \tag{22}$$

Thus, the transfers will satisfy $t_i(0, x_{-i}) = 0$ if and only if[19]

$$\sum_{y_i \in X_i(k)} \rho_i(y_i) \xi_i(y_i, x_{-i}) = 0. \tag{23}$$

Thus, there is a correspondence between ξ that satisfy (23) and participation-secure transfer rules with individual excess growths ξ.

Now, for Ξ to be the aggregate excess growth, we must have

$$\Sigma \xi(x) = \Xi(x) \tag{24}$$

for all x. Thus, the task of constructing transfers with a given aggregate excess growth reduces to constructing individual excess growths that satisfy (23) and (24).

Proposition 1. *Fix a transfer rule t and its associated individual excess growth functions ξ defined by* (21)*. Then t is given by the formula* (22)*. Moreover, t is participation secure and has aggregate excess growth Ξ if and only if ξ satisfies* (23) *and* (24)*.*

Given a balanced Ξ, we now explicitly describe the solutions to (23) and (24), which in turn define transfer rules with aggregate excess growth Ξ. For notational simplicity, we specialize to the case of $N = 2$. In Appendix B.2, we generalize all of the subsequent analysis to $N > 2$. A *balanced division of* Ξ is a pair of functions $\Xi_i : X(k) \to \mathbb{R}$ for $i = 1, 2$ such that Ξ_i is balanced and $\Xi = \Xi_1 + \Xi_2$. Any balanced division induces individual excess growths that satisfy (23) and (24), as we now explain. We interpret Ξ_i as bidder i's *initial* allocation of the aggregate excess growth. We then make "correction" to the initial excess growth to satisfy (23):

$$\xi_i(x) = \Xi_i(x) - \sum_{y_i \in X_i(k)} \Xi_i(y_i, x_j) \rho_i(y_i) + \sum_{y_j \in X_j(k)} \Xi_j(x_i, y_j) \rho_j(y_j). \tag{25}$$

[19] Brooks and Du (2020) stated and used an analogue of (23) in a continuum model where actions are non-negative real numbers. In particular, the condition (23) is key to showing that truthful reporting is an equilibrium of the strong maxmin solution constructed in that paper. A subtlety arises in the continuum model, in that there is no boundary condition at the top. Instead, the analogue of the condition (23) ensures that transfers remain bounded in the limit as x_i goes to infinity, and the transfer given by (22) converges to $\lim_{x_i \to \infty} \xi_i(x_i, x_{-i})$.

Proposition 2. *Suppose $N = 2$. If (Ξ_1, Ξ_2) is a balanced division of Ξ, then ξ given by* (25) *satisfies* (23) *and* (24)*. The corresponding transfers t defined by* (22) *are participation secure and have aggregate excess growth Ξ.*

The proof of Proposition 2 follows immediately from the definitions: Summing over x_i, the last term in the right-hand side of (25) vanishes, since Ξ_j is balanced, and the first two terms obviously cancel each other; thus, ξ defined by (25) satisfies (23). Moreover, using the assumption that $\Xi_1(x) + \Xi_2(x) = \Xi(x)$, it is easy to see that $\xi_1(x) + \xi_2(x) = \Xi(x)$ in (25) as well.

The set of solutions to (23) and (24) given by Proposition 2 is complete in the sense that if (ξ_1, ξ_2) is any solution of (23) and (24), then $(\Xi_1, \Xi_2) = (\xi_1, \xi_2)$ is clearly a balanced division, and the correction terms in (25) are all zero.

While it is simple to prove, Proposition 2 yields rich possibilities for constructing participation secure transfers with the desired aggregate excess growth. For example, we can set $\Xi_1(x) = c\Xi(x)$ and $\Xi_2 = (1-c)\Xi(x)$, where $c \in [0,1]$ is a constant, which is a balanced division whenever Ξ is balanced. Moreover, if (Ξ_1, Ξ_2) is a balanced division, then so is $(\Xi_1 + E, \Xi_2 - E)$ for any balanced function E. If E is skew-symmetric—meaning $E(x_1, x_2) = -E(x_2, x_1)$—then E will also be balanced; this follows from the fact that ρ is exchangeable, so the expectation of E over ρ is zero. Simple examples of skew symmetric functions include any odd function of the difference $x_1 - x_2$. In Section 4 we will illustrate that including a skew-symmetric E in the individual excess growth can lead to a significant simplification in the functional form of the transfers.

In summary, given (λ^*, q^*, t^*) that is feasible for (4), there will generally be many transfer rules t such that (λ^*, q^*, t) is also feasible and has the same value. For example, in characterizing max-2min transfers in the pure common value model, Brooks and Du (2020) showed that in addition to the solution given by (25) with $\Xi_i = \Xi/2$, there is a distinct solution with an especially simple form, wherein each bidder simply pays a constant price per unit, and that price depends just on the sum of the bids. This multiplicity of max-2min transfer rules, not all of which are of practical interest, presents a challenge to the study of max-2min mechanisms, and additional properties may be needed to isolate the most useful transfer rules. Going back to common values, the transfer rule in the proportional auction is characterized by the property that the aggregate transfer depends only on the aggregate action.

4 Example: constant-sum values

We now illustrate Theorem 1 with an example. In the common value model of Brooks and Du (2020), there is common knowledge that the difference in the bidders' values is zero, but there is uncertainty about the average value (i.e., the common value). We now consider what is essentially the opposite case: The average value is known, and all of the uncertainty is about the difference between the bidders' values. This is a natural model if the value of the value of the good is derived from possession of some other complementary good that is rival, in the sense that can only be possessed by one of the bidders (a government contract,

say). The bidders may have private information about the likelihood of ending up with the complementary good ex post.

To focus on a simple case, suppose that $N = 2$ and the value profiles $(1, 0)$ and $(0, 1)$ are equally likely. Thus, each bidder's ex ante expected value is $1/2$. Notice that this prior distribution does not have a full support, so we cannot conclude on the basis of Theorem 1 that the optimal value of the max-2min program (4) asymptotically converges to that of the min-2max program (3). However, it does hold for nearby models where the probability of the value profiles $(1, 1)$ and $(0, 0)$ is positive but arbitrarily small. We will prove that the asymptotic convergence holds when $(1, 1)$ and $(0, 0)$ occur with probability zero by explicitly constructing min-2max information structures and max-2min mechanisms.

4.1 Min-2max information structures

Since it is possible that bidders have no information about their values, no mechanism can guarantee more revenue than the ex ante expected value. In particular, for any k, there is a feasible solution to (3) in which $\sigma(x, v) = \rho(x)/2$ for all (x, v), so that $w(x) = 1/2$ for all x, and hence $\gamma(x) = \rho(x)/2$ is feasible, and the value of the solution is $1/2$. As a result, $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \leqslant 1/2$ for all k.

Moreover, for any information structure, there is a mechanism that guarantees the seller expected profit of $1/2$: Fix an information structure $\mathcal{I} = (S, \sigma)$ and let $w_i(s)$ be the interim expectation of bidder i's value given the signal profile s. Then clearly, $w_1(s) + w_2(s) = 1$, so there is at least one bidder for whom $w_i(s) \geqslant 1/2$. Consider the direct mechanism that allocates the good to whichever bidder has $w_i(s) \geqslant 1/2$ (breaking ties arbitrarily when $w_1(s) = w_2(s) = 1/2$), and charges the bidder a price of $1/2$ whenever they are allocated the good. It is straightforward to verify that this mechanism is incentive compatible and individually rational, and hence can be made participation secure. And since the good is always allocated, expected revenue is $1/2$.

Thus, we conclude that the value of the min-2max program is $1/2$ for all S, and no information is a min-2max information structure.

4.2 Posted prices and discussion mechanisms

As stated above, because this example violates the full support hypothesis, Theorem 1 does not immediately imply that $\lim_{k\to\infty} \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) = 1/2$. Even so, we shall see that there are participation-secure mechanisms that guarantee the seller profit arbitrarily close to $1/2$. The construction of such mechanisms is a subtle matter, though, as we now explain.

A natural first guess is that the seller can guarantee revenue of $1/2$ by simply posting a price of $p = 1/2 - \epsilon$ for ϵ sufficiently small (so as to break ties in favor of purchasing). Such mechanisms would be approximately optimal at the min-2max information structure. However, if the seller posts a price of p, there are other information structures and equilibria in which the probability of not making a sale (and hence not generating revenue) is bounded away from zero. In particular, suppose that each bidder's set of signals is $S_i = \{0, 1\}$, the signal matches the true value with probability $3/4$ and mismatches with probability $1/4$, and the noise is independent conditional on the values. When ϵ is sufficiently small, there

is an equilibrium of the posted price mechanism with this information structure in which bidders purchase if and only $s_i = 1$. This follows from the fact that conditional on a signal $s_i = 0$ and asking to purchase the good, the posterior probability that $v_i = 1$ is only 1/4, so that the expected utility from buying the good is

$$\frac{1}{4}\left(\frac{3}{4} + \frac{1}{4}\frac{1}{2}\right)(1-p) + \frac{3}{4}\left(\frac{1}{4} + \frac{3}{4}\frac{1}{2}\right)(0-p) = -\frac{1}{8} + \frac{11}{16}\epsilon,$$

which is negative when $\epsilon < 2/11$. (In this calculation, we have accounted for the fact that bidders only win half of the time when they both ask to buy.)

So, posted prices are not max-2min mechanisms. Nonetheless, we can argue heuristically that there should exist mechanisms that guarantee profit close to 1/2 in all equilibria: Suppose the seller offers the good at a price of $p = 1/2 - \epsilon$. The seller then lets the bidders "discuss" what they think the good is worth, for as long as they like, until they reach a consensus about whether or not, say, bidder 1's interim expected value is greater than the price. Aumann's agreement theorem implies that if the bidders have common knowledge of their beliefs about whether $v_1 > p$, then they must have the same belief. This results in one of two outcomes: (i) both bidders believe that bidder 1's interim expected value is strictly greater than p, in which case bidder 1 will strictly prefer to purchase the good); or (ii) both bidders believe that bidder 1's interim expected value is less than p, in which case they agree that bidder 2's value must be at least $1 - p > p$ (since the sum of the values is one), in which case bidder 2 will strictly prefer to purchase. Either way, one of the bidders strictly prefers to purchase the good, so that the seller earns revenue of p.

This argument is conceptually satisfying but it is not obvious how to translate it into an actual mechanism. How should the conversation between the bidders be structured? Can it be implemented with finite mechanisms? How do we make sure the bidders get "close enough" to common knowledge in every equilibrium? Instead of dealing with these various issues, we will simply construct feasible solutions for the program (4) that have value close to 1/2. In fact, we will construct a feasible solution (λ, Ξ, q) for $(4')$ such that the value is close to 1/2, in which case we know that a solution with the same allocation and value exists for (4).

4.3 Max-2min mechanism

Let $\lambda(v) \equiv 1/2$ for all v. Fix a positive number m, and define

$$q_i(x) \equiv \begin{cases} 1 & \text{if } x_i > x_j + m; \\ 0 & \text{if } x_i < x_j - m; \\ \frac{x_i - x_j + m}{2m} & \text{otherwise} \end{cases} \tag{26}$$

and

$$\begin{aligned} \Xi(x) &\equiv \min_v v \cdot \nabla^+ q(x) - \lambda(v) \\ &= \min_{i=1,2} \nabla_i^+ q(x) - 1/2. \end{aligned}$$

Note that

$$\nabla_i^+ q(x) = \begin{cases} \frac{k-1}{k}\frac{1}{2m} & \text{if } m > x_i - x_j \geqslant -m \text{ and } x_i < k; \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\Xi(x) = -1/2 + \begin{cases} \frac{k-1}{k}\frac{1}{2m} & \text{if } |x_1 - x_2| < m \text{ and } \max(x_1, x_2) < k; \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Now, when k is large, x converges in distribution to independent exponential, so $x_1 - x_2$ converges to Laplace, and $|x_1 - x_2|$ converges to exponential. Thus, when k is large we have

$$\begin{aligned} \sum_{x \in X(k)} \rho(x)\Xi(x) &\approx -1/2 + \frac{1}{2m}\int_{y=0}^{m} \exp(-y)dy \\ &= -1/2 + \frac{1 - \exp(-m)}{2m}. \end{aligned}$$

The limit profit guarantee associated with this mechanism is therefore arbitrarily close to $(1 - \exp(-m))/2m$. Using L'Hôpital's rule, we find that

$$\lim_{m \to 0} \frac{1 - \exp(-m)}{2m} = \lim_{m \to 0} \frac{\exp(-m)}{2} = \frac{1}{2}.$$

Hence, by first taking k large and then m small, the seller can guarantee profit arbitrarily close to 1/2. We conclude that the max-2min program has a value of 1/2 in the limit as $k \to \infty$ and as $m \to 0$. In this limit, the good is essentially allocated to whichever bidder has the highest action.

Note that for finite k, the function Ξ given by (27) is not balanced. But it is straightforward to modify the solution by setting C equal to the expectation of Ξ, replacing λ and Ξ with $\lambda + C$ and $\Xi - C$, respectively. Then $\Xi - C$ is balanced, and hence by Lemma 4 and Proposition 1, there exist participation secure transfers t with aggregate excess growth $\Xi - C$, so that $(\lambda + C, q, t)$ is feasible for (4) and has value close to $\Pi^* = 1/2$.

Finally, we relate the optimal Ξ to the transfers in the max-2min mechanism. As before, we first take $k \to \infty$ and then $m \to 0$ in our computation of the transfers using Propositions 1 and 2. In Appendix B.3 we show that the balanced division $\Xi_1(x) = \Xi_2(x) = \Xi(x)/2$ leads to fairly complicated transfers (equation (41)). However, by adding an odd function to the division ($\Xi_i(x) = \Xi(x)/2 - \text{sign}(x_i - x_j)/4$, where $\text{sign}(z)$ is the sign of z), we get the following transfers in the limit:

$$t_i(x) = \begin{cases} 0 & x_i < x_j; \\ 1/4 & x_j = x_i; \\ 1/2 & x_i > x_j, \end{cases}$$

when $x_j > 0$, and

$$t_i(x) = \begin{cases} 0 & x_i = 0; \\ 1/4 & x_i > 0, \end{cases}$$

when $x_j = 0$. Thus, as $m \to 0$ the good is allocated to the highest bidder for a posted price of 1/2!

4.4 Extensions

The example can be easily generalized to the case where the prior μ is supported on value profiles such that $v_1 + v_2 = 2\hat{v}$ for some constant $\hat{v}$, and the two bidders have the same ex ante expected value $\hat{v}$. (We do not need to assume that μ is exchangeable.) We can proceed with the same allocation q as before, $\lambda(v) = \hat{v}$ for all v,

$$\begin{aligned}\Xi(x) &= 2\hat{v} \min_{i=1,2} \nabla_i^+ q(x) - \hat{v} \\ &\leqslant \min_v v \cdot \nabla^+ q(x) - \lambda(v)\end{aligned}$$

for all x, and multiplying the transfers by a factor of $2\hat{v}$. Our constructed solution (λ, Ξ, q) is still feasible for (4′), and the mechanism remains max-2min optimal.

More interesting is the generalization to more than two bidders, while maintaining the hypothesis that $\Sigma v = C$ for every v in the support of μ. In this case, it is straightforward to extend our characterization of the min-2max program, which has value C/N. The critical step in showing that this is also the limit value of the max-2min program is to construct an allocation with the property that

$$\sum_{x \in X(k)} \rho(x) \min_{i=1,\ldots,N} \nabla_i^+ q(x) \approx 1/N.$$

Simulations indicate that such an allocation does exist for $N = 3$, and we conjecture that it exists for $N > 3$ as well.

4.5 Further examples

In the Online Appendix, we report additional numerical examples. First, we give an example with values that are perfectly correlated, but where one bidder's value is always larger by a fixed amount. In this case, max-2min mechanisms distort the allocation towards the higher-value bidder, and the lower-value bidder is only allocated the good when the low-value bidder's action is relatively low. We also give an example in which values are independent uniform. In this case, when the good is fully allocated, the allocation seems to only depend on the difference in the bidders' actions. The Online Appendix also contains numerical examples that illustrate the extensions described in the next section.

5 Further results

We now present further theoretical results. First, we state a simple corollary of Theorem 1, which is a "strong minimax theorem" for the zero-sum game between the seller and an adversary, and we relate our results to the strong maxmin solution concept proposed in Brooks and Du (2020). We then discuss several ways in which Theorem 1 can be extended

to cases where there are extra feasibility constraints on the allocation, multiple goods, and ambiguity about the value distribution. We then discuss the rate of convergence to the profit guarantee, robustness of the profit guarantee to misspecification of the value distribution, and other objectives besides profit.

5.1 Strong maxmin solutions and the strong minimax theorem

As mentioned in the introduction, Brooks and Du (2020) characterize analogues of the the max-2min and min-2max programs in a setting with infinitely many actions and signals and pure common values. When the action and signal space are infinite, equilibrium existence is not guaranteed, and care has to be taken that bounds on equilibrium profit in an information structure or mechanism are not vacuous. To finesse the equilibrium existence issue, we defined and worked with a new solution concept which we termed a *strong maxmin solution*, which is a triple $(\mathcal{M}, \mathcal{I}, b)$ such that b is an equilibrium of $(\mathcal{M}, \mathcal{I})$, and relative to that equilibrium, neither the seller nor the adversary choosing the information structure can deviate in a way that moves profit in their preferred direction, even if the deviator can select the equilibrium.

With finite action and signal spaces, an equilibrium always exists, but it may not be possible to attain the profit guarantee exactly. We now formulate a solution concept that is appropriate to this setting and analogous to the strong maxmin solution. We then present an existence result.

A pair $(\mathcal{M}, \mathcal{I})$ is a (Π, ϵ)-*strong maxmin solution* if

(i) for every $\mathcal{I}'$ and $b' \in B(\mathcal{M}, \mathcal{I}')$, $\Pi(\mathcal{M}, \mathcal{I}', b') \geqslant \Pi - \epsilon$;

(ii) for every $\mathcal{M}'$ and $b' \in B(\mathcal{M}', \mathcal{I})$, $\Pi(\mathcal{M}', \mathcal{I}, b') \leqslant \Pi + \epsilon$.

Thus, $\mathcal{M}$ guarantees the seller a profit of at least $\Pi - \epsilon$ in any information structure and any equilibrium, and $\mathcal{I}$ guarantees that the seller's profit is at most $\Pi + \epsilon$ in any mechanism and any equilibrium. Note that (i) and (ii) together imply that every equilibrium in $(\mathcal{M}, \mathcal{I})$ has profit in $[\Pi - \epsilon, \Pi + \epsilon]$.

An immediate corollary of Theorem 1 is the following:

Theorem 2. *Suppose $\mu(v) > 0$ for all $v \in V$. Then,*

$$\Pi^* \equiv \inf_{\mathcal{I} \in \mathbf{I}} \sup_{\mathcal{M} \in \mathbf{M}} \sup_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b) = \sup_{\mathcal{M} \in \mathbf{M}} \inf_{\mathcal{I} \in \mathbf{I}} \inf_{b \in B(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, b). \tag{28}$$

Moreover, there exists a (Π, ϵ)-strong maxmin solution for every $\epsilon > 0$ if and only if $\Pi = \Pi^*$.

Proof of Theorem 2. The left-hand side of (28) is equal to the inf over all finite signal spaces S of $\Pi^{\text{MIN-2MAX}}(S)$. By Lemma 1, $\Pi^{\text{MIN-2MAX}}(X(k))$ is a decreasing sequence, so that the left-hand side of (28) is less than or equal to the limit as k goes to infinity of $\Pi^{\text{MIN-2MAX}}(k)$, which by Theorem 1 is Π^*. A similar argument shows that the right-hand side of (28) is at least Π^* as well. The theorem then follows from Lemma 1, which implies that the left-hand side of (28) is greater than the right-hand side.

Now, for any $\epsilon > 0$, there exists k such that $\overline{\Pi}^{\text{MIN}-\text{2MAX}}(k) \leqslant \Pi^* + \epsilon$ and $\underline{\Pi}^{\text{MAX}-\text{2MIN}}(k) \geqslant \Pi^* - \epsilon$. For this k, let $\mathcal{I} = (X(k), \sigma)$ for any σ that is part of an optimal solution to (3) and let $\mathcal{M} = (X(k), q, t)$ for any (q, t) that are part of an optimal solution to (4). It is immediate from Theorem 1 that $(\mathcal{M}, \mathcal{I})$ is a (Π^*, ϵ)-strong maxmin solution.

Finally, let $\Pi \neq \Pi^*$ and suppose that $(\mathcal{M}, \mathcal{I})$ be a (Π, ϵ)-strong maxmin solution for $\epsilon < |\Pi - \Pi^*|$. Then either $\Pi - \epsilon > \Pi^*$, in which case $\mathcal{M}$ guarantees profit greater than Π^* in all information structures and equilibria, or $\Pi + \epsilon < \Pi^*$, in which case $\mathcal{I}$ guarantees profit less than Π^* for all mechanisms and equilibria. In the first case, we contradict the convergence of $\overline{\Pi}^{\text{MIN}-\text{2MAX}}(k)$ to Π^*, and in the second case, we contradict the convergence of $\underline{\Pi}^{\text{MAX}-\text{2MIN}}(k)$ to Π^*. □

Theorem 2 is essentially a "strong minimax theorem" for the zero-sum game in which the seller chooses the mechanism to maximize profit and information is chosen adversarially to minimize profit. It is important to note, however, that this is not quite a game, because for a given $(\mathcal{M}, \mathcal{I})$, there may be multiple equilibria with different profit levels, and therefore the standard results on zero-sum games do not apply. In the left-hand side of (28), we have effectively selected the seller's preferred equilibrium, and in the right-hand side, we selected the profit-minimizing equilibrium. The theorem therefore implies that the values of these programs would coincide regardless of how an equilibrium is selected. Equivalently, we could consider the collection of zero-sum games that are parametrized by the choice of equilibrium selection rule (as a function of the mechanism and information structure). Then any (Π, ϵ)-strong maxmin solution is an ϵ-equilibrium of all of the zero-sum games with different equilibrium selection rules.

5.2 Extensions

We proved our main results in a benchmark setting where there is a single unit of a good for sale, and the joint distribution of bidders' values is known. In fact, our methodology can be easily extended in a number of directions. Appendix B.1 contains a detailed discussion of these variations of our model and numerical examples. We here summarize the key points.

Feasibility constraints

The model can be generalized to allow for extra constraints on the allocation. For example, it could be that bidder 1 demands one unit of the good, but bidder 2 demands only half a unit. Or it could be that there is a constraint on the share of the good that can be allocated to a subset of the bidders. The addition of such constraints on the allocation introduces new variables in (3), which are multipliers on the extra feasibility constraints, and the extra constraints themselves appear in (4). But these new variables and constraints have very little impact on the rest of our argument: showing the asymptotic equivalence of (3) and (4) reduces to showing that the difference between updward and downward constraints is small when k is large. This step goes through largely as before, as long as the feasibility constraints are *downward closed*, in the sense that if q is feasibility and $q' \leqslant q$, then q' is also feasible.

Multiple goods

We can allow for multiple goods to be auctioned simultaneously, where each bidder demands a single unit of each good and values are additive. The problem of auctioning the goods is linked through the information structure, which now consists of sets of signals and a joint distribution over bidders' signals and bidders' values for all goods. We can proceed as before, fixing an arbitrary order on signals, taking the local relaxations of the min-2max and max-2min programs, and fixing multipliers on local incentive constraints, to arrive at generalizations of the restricted programs (3) and (4). In Appendix B.1, we explain why the inequality (20) will hold for separately for each good, which is the key to proving convergence.

An interesting special case is when the distribution of values is *exchangeable across goods*, meaning that if the two value profiles v and v' are related by permuting bidder i's values for the different goods, then the value profiles are equally likely. In this case, it is without loss to restrict attention to mechanisms in which the seller only offers the grand bundle, and to information structures in which bidders only receive information about the value of the grand bundle. The reason is as follows. Clearly, if we restrict the seller to only offering the grand bundle in the max-2min program, then it is without loss to consider information structures that only give information about the value of the grand bundle. In the other direction, suppose the value distribution is exchangeable across goods and the information structure is only directly informative about the value of the grand bundle. Under such information, and supposing that the value distribution is exchangeable across goods, the bidders will have the same interim expected value for each good, so it is without loss to restrict the seller to mechanisms that in which the allocation is the same for all goods, i.e., the seller only offers the grand bundle.

In independent and concurrent work, Deb and Roesler (2021) studied the design of informationally robust optimal auctions in the special case of a single bidder. They also conclude that when values are exchangeable across goods, there is a max-2min mechanism in which the seller only offers the grand bundle. In the Online Appendix, we give numerical examples in which the exchangeability hypothesis fails, and max-2min mechanisms offer more than just the grand bundle.

Ambiguous value distribution

In our baseline model, we assumed that the seller knows the joint distribution of the bidders' values. Of course, given that the seller already has a concern for robustness with respect to information and equilibrium, they might naturally also be concerned about robustness with respect to the distribution of values. In Section 5.4, we will argue that the profit guarantee of max-2min mechanisms is robust to misspecification of the value distribution. But another way to proceed is to build that concern for robustness directly into the model, for example, by fixing only the marginal distribution of each bidder's value, and taking a worst-case over joint distributions of bidders' values.[20] In the program (3), this is equivalent

[20]Carroll (2017) studies a robust multiple-good monopoly problem where the designer knows marginal distributions but takes a worst case over the joint distribution. In contrast, we model the sale of a single unit to multiple bidders, where there is ambiguity about the joint distribution of bidders' values. Moreover,

to replacing the marginal constraint in the definition of an information structure with the new constraints

$$\sum_{s,v_{-i}} \sigma(s, v_i, v_{-i}) = \mu_i(v_i) \ \forall i, v_i,$$

where μ_i is the prior distribution of bidder i's value. In (4), this is equivalent to imposing the functional form restriction that $\lambda(v) = \sum_i \lambda_i(v_i)$ for some functions $\lambda_i : V_i \to \mathbb{R}$. This additive-separability condition does not interact with our convergence argument, although now the full-support assumption is that $\mu_i(v_i) > 0$ for all i and v_i.

In Brooks and Du (2021), we study a version of this model with binary values $V = \{0, \overline{v}\}$. This is equivalent to assuming that we know only the average value of each bidder and an upper bound on each bidder's value. In that paper, we show that when the bidders have the same expected value, the worst-case correlation structure is one in which values are perfectly correlated. In other words, the model reduces to one of pure common values, previously studied by Brooks and Du (2020). In particular, the proportional auctions, which were found to be max-2min optimal when values are common, remain max-2min optimal even when the correlation structure is unknown. When bidders have different means, the max-2min allocation has the form of a weighted proportional rule.

Alternatively, one could eliminate marginal constraints altogether, and represent the seller's ambiguity about the value distribution using variational preferences as in Hansen and Sargent (2001) and Maccheroni, Marinacci, and Rustichini (2006). In particular, we could fix a function $\lambda : V \to \mathbb{R}$, and add a penalty to the seller's payoff, which is the expectation of λ under σ. In Appendix B.1, we explain how our main results extend with this model of preferences.

This list of extensions is not meant to be exhaustive. We suspect that our methods can be further generalized, such as to environments with different objectives or richer preferences for bidders. This is an important direction for future work.

5.3 Rate of convergence

Our next result characterizes the rate of convergence of the profit bounds to Π^*:

Proposition 3. *Suppose that $\mu(v) > 0$ for all $v \in V$. For all $k \geqslant 1$,*

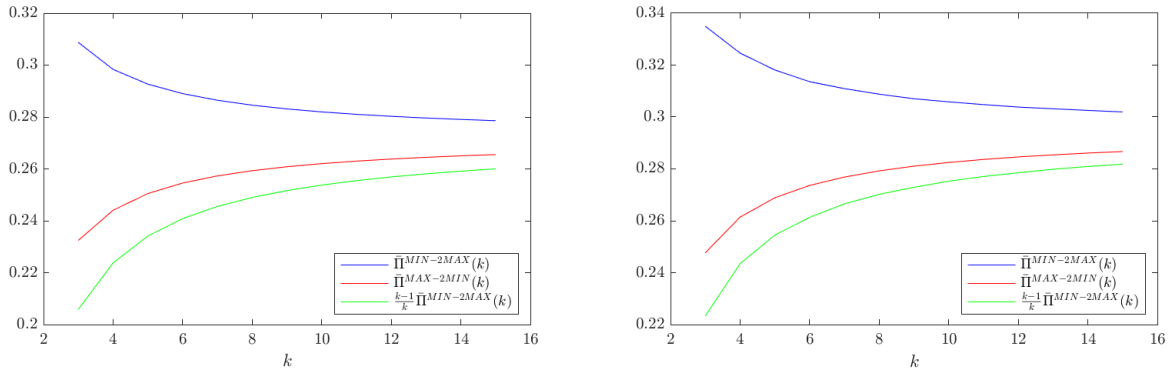
$$\left| \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \right| \leqslant \frac{\overline{v}}{k} + o(1/k).$$

Hence, $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$ and $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$ converge to Π^* at a rate of $1/k$.

Proof of Proposition 3. Theorem 1 shows that $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \geqslant \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$. The proof of Lemma 6 shows that

$$\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \geqslant \frac{k-1}{k+C} \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \left(1 - \frac{1}{k}\right)^{k^2-1} \left(\frac{k-1}{k+C} k + k - 1\right) N\overline{v}.$$

Carroll (2017) assumes that the agent knows their ex post values, and just varies the correlation structure, whereas our model incorporates ambiguity about bidders' information about their own values.



(a) Pure common values

(b) Independent values

Figure 1: Rate of convergence.

for a constant C. Hence,

$$\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) - \underline{\Pi}^{\text{MAX}-2\text{MIN}}(k) \leqslant \frac{1+C}{k+C}\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) + \left(1 - \frac{1}{k}\right)^{k^2-1}\left(\frac{k-1}{k+C}k + k - 1\right) N\overline{v}.$$

Since $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k) \leqslant \overline{v}$, this immediately gives the first result. The second result follows immediately from the fact that $\Pi^* \in [\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k), \overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)]$ for all k. □

We illustrate this rate of convergence with two examples, depicted in Figure 1. Both examples are for $N = 2$ and values in a grid $V = \{0, 0.1, \ldots, 1\}$. In the left panel, bidders have a pure common value that is uniform $\{0, 0.1, \ldots, 1\}$, i.e., μ is concentrated on the diagonal. In the right-hand panel, values are independent uniform on V. In each case, the blue and red lines are $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$ and $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$, respectively. Note that only the independent example satisfies the full-support hypothesis of Proposition 3. Nonetheless, in both cases the green line is $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)(k-1)/k$ and is always below the red line, consistent with the theoretical rate of convergence.

We note that if $(\gamma^*, \sigma^*, w^*)$ is a solution to (3), then $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$ is only an upper bound on maximum profit of the information structure $(X(k), \sigma^*)$ across all mechanisms and equilibria, which is in turn only an upper bound on the value of (1). It is logically possible that both optimal profit in solutions to (3) and $\Pi^{\text{MAX}-2\text{MIN}}(X(k))$ converge to Π^* at a faster rate. A corresponding statement applies to solutions to (4) and $\Pi^{\text{MAX}-2\text{MIN}}(X(k))$.

5.4 Robustness to fundamentals

An important feature of the approximate max-2min mechanisms, in addition to their optimal worst-case performance, is that we can bound their performance even when the value distribution for which they were derived is misspecified.

To develop this result, we need the following lemma, which asserts that any *feasible* solution to the programs (3) or (4) has a corresponding bound on equilibrium expected profit.

Lemma 7. *Fix $k \geqslant 1$. Suppose (γ, σ, w) is a feasible solution to (3), and let $\mathcal{I} = (X(k), \sigma)$. Then*

$$\sup_{\mathcal{M}' \in \mathbf{M}} \sup_{b \in B(\mathcal{M}', \mathcal{I})} \Pi(\mathcal{M}', \mathcal{I}, \beta) \leqslant \sum_{x \in X(k)} \gamma(x).$$

Suppose (λ, q, t) is a feasible solution to (4), and let $\mathcal{M} = (X(k), q, t)$. Then

$$\inf_{\mathcal{I}' \in \mathbf{I}} \inf_{b \in B(\mathcal{M}, \mathcal{I}')} \Pi(\mathcal{M}, \mathcal{I}', \beta) \geqslant \sum_{v \in V} \lambda(v)\mu(v).$$

Proof of Lemma 7. This is an immediate implication of the proofs of Lemmas 2 and 3. In particular, the program (3) is obtained by taking the dual of the inner maximization over mechanisms and equilibria from (1), so that any feasible solution to that dual provides an upper bound on the value of the primal, meaning that it provides an upper bound on profit under any mechanism and equilibrium. Similarly, we obtained (4) by taking the dual of the inner minimization program, and any feasible solution to the dual provides a lower bound on profit in the primal program. □

We can now formalize the bounds for a misspecified prior, which generalizes Proposition 7 of Brooks and Du (2020):

Proposition 4. *Suppose that (λ, q, t) is a feasible solution to (4), and extend the domain of λ to all of $\mathbb{R}^N_+$ according to*

$$\lambda(v) = \min_{x \in X(k)} \left[\Sigma t(x) + v \cdot \nabla^+ q(x) - \nabla^+ \cdot t(x) \right]$$

Then for all $\mu' \in \Delta(\mathbb{R}^N_+)$ with finite support, revenue in any information structure and equilibrium is at least

$$\sum_{v \in \mathbb{R}^N_+} \lambda(v)\mu'(v). \tag{29}$$

In particular, the bound (29) holds for an optimal solution (λ^*, q^*, t^*).

Proof of Proposition 4. Suppose that the prior is μ'. Then clearly (λ, q, t) is a feasible solution for the program (4), where we replace μ with μ'. From Lemma 7, we conclude that (29) is a lower bound on equilibrium profit. □

5.5 Other objectives

The focus of our analysis until this point has been expected profit. One may ask: Does our model have interesting implications for other welfare objectives, in particular total surplus? As the following proposition shows, the answer to this particular question is essentially no. Given a mechanism $\mathcal{M} = (A, q, t)$, information structure $\mathcal{I} = (S, \sigma)$, and strategy profile b, let

$$TS(\mathcal{M}, \mathcal{I}, b) = \sum_{s \in S} \sum_{v \in V} \sum_{a \in A} v_i q_i(a) b(a|s) \sigma(s, v)$$

denote the resulting expected total surplus. Let

$$\underline{TS} = \max_{i=1,\ldots,N} \sum_{v \in V} v_i \mu(v)$$

denote the highest ex ante expected value among the bidders.

Proposition 5.

$$\sup_{\mathcal{M} \in \mathbf{M}} \inf_{\mathcal{I} \in \mathbf{I}} \inf_{b \in BNE(\mathcal{M},\mathcal{I})} TS(\mathcal{M},\mathcal{I},b) = \inf_{\mathcal{I} \in \mathbf{I}} \sup_{\mathcal{M} \in \mathbf{M}} \sup_{b \in BNE(\mathcal{M},\mathcal{I})} TS(\mathcal{M},\mathcal{I},b) = \underline{TS}.$$

Proof of Proposition 5. Let i be the index of a bidder with the highest ex ante expected value. Then clearly a feasible mechanism is $q_i(a) = 1$ and $q_j(a) = 0$ for all a and $j \neq i$, and $t_j(a) = 0$ for all a and j. This mechanism is guaranteed to generate $\underline{TS}$ regardless of the information structure and equilibrium. On the other hand, under the degenerate information structure in which each bidder has a single signal, then total surplus in any mechanism and equilibrium must be less than $\underline{TS}$. □

Thus, in order to obtain non-degenerate results for social welfare, one either needs to modify the objective (such as by using min-max regret) or by imposing restrictions on the set of information structures so that the domain of minimization does not include a least informative information structure.

6 Conclusion

This paper has developed new tools for informationally robust optimal auction design. The focus of our inquiry has been the min-2max and max-2min programs, which provide complementary characterizations of the profit that a seller can guarantee uniformly across all models of bidders' information. Our results show that these programs have the same value, so that a seller can guarantee themselves the same level of profit, regardless of whether the mechanism can directly condition on the information structure and regardless of which equilibrium is selected. Moreover, it is without loss to restrict attention to solutions that are characterized by binding local constraints, in which case the max-2min and min-2max programs become linear optimization problems. Our results facilitate both numerical computation and analytical characterization of approximate max-2min mechanisms and min-2max information structures. We now discuss a number of promising directions for future research.

- The arguments employed in this paper are non-constructive. Beyond a handful of cases which have been solved, relatively little is known about the detailed form of max-2min mechanisms and min-2max information structures. A more complete analysis in special cases could lead to the discovery of novel auction formats that are of independent interest, such as the proportional auction.

- For analytical simplicity, we have restricted attention to finite mechanisms and information structures, and in simulations, the profit guarantee is only obtained asymptotically. In contrast, Brooks and Du (2020) work with mechanisms and information

structures for which actions and signals are continuous. Can the results of this paper be reformulated in a continuous model, so that the profit guarantee is attained exactly? Can our approximate duality result be strengthened to exact duality? Can our existence result for approximate strong maxmin solutions be strengthened to existence of strong maxmin solutions, in the sense of Brooks and Du (2020)?

- We have described several straightforward extensions. Can our approach be extended further, for example, to environments with more complicated feasibility constraints and/or richer preferences over allocations? Can the results be generalized in a non-trivial manner to other objectives besides profit maximization?

- Perhaps most importantly, our theory places no restrictions on bidders' information, beyond the common prior assumption and the known marginal distribution of bidders' values. While the present model does address robustness to model misspecification, the bounds obtained with with no restrictions on information may be too conservative. We predict that this theory will become more useful as it incorporates more flexible restrictions on bidders' information that allow us to explore the middle ground between this theory and the standard model.

We conclude with a discussion of possible interpretations of our results. Our characterization of the max-2min program can be understood literally as predicting the choices of a seller who evaluates each mechanism by its worst-case profit across all information structures and equilibria. We do not believe that real-world auction designers have such extreme preferences. At the same time, we suspect that designers in a practical setting may be unable or unwilling to commit to a single model of information and a single equilibrium as the correct description of behavior, as required by the classical Bayesian auction design paradigm. Our view is that the truth is somewhere in between: Designers may know some features of bidders' information without being able to give a complete description. Of course, ambiguity about bidders' information may be accompanied by distinct concerns about the complexity of the mechanism and/or the accuracy of the equilibrium prediction. It is beyond our present abilities to incorporate all such concerns into the theory of optimal auctions. We can, however, ask what mechanisms are robust to ambiguity about bidders' information in an extreme sense, provided we are still willing to accept the common prior and Bayes Nash equilibrium as an as-if description of behavior. Our results show that the seller need not have the bidders explicitly communicate their higher-order beliefs in order to attain the optimal profit guarantee.

In our view, the greatest promise of this approach is that it may lead to the discovery of new auction designs, such as the proportional auction, that are compelling both for their optimal worst-case performance as well as for their simplicity.[21] The worst-case performance of a mechanism is, in a sense, a measure of how "safe" it is. To be sure, it is just one of many criteria that might be considered in applied auction design. For example, one may also weigh how the auction performs on particular, benchmark information

[21]The worst-case analysis naturally leads to a great deal of structure on information and mechanisms, which we view as being relatively "simple," at least compared to the benchmark of full surplus extraction mechanisms in correlated type spaces (Crémer and McLean, 1988; McAfee et al., 1989): As we show, there always exist approximate min-2max information structures with independent signals.

structures, such as affiliated values. Importantly, there need not be conflict between these criteria: when values are common and the number of bidders is large, the profit guarantee is approximately the entire surplus, so that max-2min mechanisms are near optimal in all information structures (Du, 2018; Brooks and Du, 2020). This will not always be the case, however, and an important task for future work is to evaluate max-2min auctions on particular information structures and under different solution concepts. Such analyses will lead to a more balanced view of the merits and demerits of the max-2min auctions, and the tradeoff between informational robustness and Bayesian optimality.

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A Omitted proofs

Proof of Lemma 1. Fix an $\epsilon > 0$, let $\mathcal{M} \in \mathbf{M}(A)$ be a mechanism such that the infimum profit across information structures and equilibria is at least $\Pi^{\text{MAX-2MIN}}(A) - \epsilon$. Let $\mathcal{I} \in \mathbf{I}(S)$ be an information structure such that the supremum profit across mechanisms and equilibria is at most $\Pi^{\text{MIN-2MAX}}(S) + \epsilon$. Thus, for any equilibrium $b \in B(\mathcal{M}, \mathcal{I})$, $\Pi^{\text{MIN-2MAX}}(S) + \epsilon \geqslant \Pi(\mathcal{M}, \mathcal{I}, b) \geqslant \Pi^{\text{MAX-2MIN}}(A) - \epsilon$, thus showing that $\Pi^{\text{MIN-2MAX}}(S) \geqslant \Pi^{\text{MAX-2MIN}}(A) - 2\epsilon$. Since ϵ is arbitrary, we conclude that $\Pi^{\text{MIN-2MAX}}(S) \geqslant \Pi^{\text{MAX-2MIN}}(A)$.

We now prove the second part. For $\epsilon > 0$, let $\mathcal{M} = (A, q, t)$ be a mechanism such that infimum profit across information structures and equilibria is at least $\Pi^{\text{MAX-2MIN}}(A) - \epsilon$. Since $|A'_i| > |A_i|$, there exists an onto mapping $f_i : A'_i \to A_i$ for each i. Let $f : A' \to A$ be the product mapping. We define $q'(a') = q(f(a'))$ and $t'(a') = t(f(a'))$, and let $\mathcal{M}' = (A', q', t')$. It is clear that for every $\mathcal{I}$ and Π, there is an equilibrium of $(\mathcal{M}, \mathcal{I})$ with profit Π if and only if there is an equilibrium of $(\mathcal{M}', \mathcal{I})$ with profit Π: For given the former, we can construct a profit-equivalent equilibrium of the latter by selecting a single action in $f_i^{-1}(a_i)$ to be played instead of a_i, and given the latter, we can construct a profit equivalent equilibrium in which the action $f_i(a'_i)$ is played instead of a'_i. Thus, $\Pi^{\text{MAX-2MIN}}(A') \geqslant \Pi^{\text{MAX-2MIN}}(A) - \epsilon$, and since ϵ is arbitrary, we have $\Pi^{\text{MAX-2MIN}}(A') \geqslant \Pi^{\text{MAX-2MIN}}(A)$. The proof for $\Pi^{\text{MIN-2MAX}}$ is analogous and is omitted. □

We now develop the proof of Lemma 6. Let (λ^*, Ξ^*, q^*) be an optimal solution of (3-D). Without loss of optimality, we can assume that

$$\sum_{v \in V} \mu(v)\lambda^*(v) = \overline{\Pi}^{\text{MIN-2MAX}}(k)$$

and

$$\sum_{x \in X(k)} \rho(x)\Xi^*(x) = 0.$$

Lemma 8. *Suppose $\mu(v) > 0$ for all $v \in V$. Then $|\lambda^*(v)| \leqslant \max_{v' \in V} \frac{\overline{v}}{\mu(v')}$ for all $v \in V$ and all k.*

Proof of Lemma 8. We first show that for all k and $v \in V$, $\lambda^*(v) \leqslant \overline{v}$. For the sake of contradiction, suppose not, i.e., there exist some k and v' such that $\lambda^*(v') > \overline{v}$. Consider the dual of (3-D) where we fix λ^*:

$$\begin{aligned}
&\min_{\gamma : X(k) \to \mathbb{R}_+,\, \sigma : X(k) \times V \to \mathbb{R}_+} \sum_{x \in X(k)} \gamma(x) - \sum_{v \in V, x \in X(k)} \lambda^*(v)(\sigma(v, x) - \mu(v)) \\
&\text{s.t. } \gamma(x) \geqslant \begin{cases} \sum_{v \in V} v_i k[\sigma(x, v) - \sigma(x_i + 1/k, x_{-i}, v)] & \text{if } x_i < k - 1/k, \\ \sum_{v \in V} v_i [k\sigma(x, v) - \sigma(x_i + 1/k, x_{-i}, v)] & \text{if } x_i = k - 1/k, \\ \sum_{v \in V} v_i \sigma(x, v) & \text{if } x_i = k, \end{cases} \\
&\qquad \forall i, x; \\
&\sum_{v \in V} \sigma(x, v) = \rho(x) \ \forall x,
\end{aligned} \tag{30}$$

Let $\overline{\sigma}(x,v) = \rho(x)\delta_{v'}(v)$ and $\overline{\gamma}(x) = \rho(x)\max_i v'_i$. It is easy to check that $(\overline{\sigma},\overline{\gamma})$ is feasible for the above program and obtains an objective strictly less than $\max_i v'_i - \overline{v} + \overline{\Pi}^{\text{MIN-2MAX}}(k) < \overline{\Pi}^{\text{MIN-2MAX}}(k)$. This is an contradiction since the optimal value of (30) is $\overline{\Pi}^{\text{MIN-2MAX}}(k)$.

Since $\lambda^*(v) \leqslant \overline{v}$ and $\sum_v \mu(v)\lambda^*(v) = \overline{\Pi}^{\text{MIN-2MAX}}(k)$, we must have $\lambda^*(v) \geqslant (\overline{\Pi}^{\text{MIN-2MAX}}(k) - \overline{v})/\mu(v) \geqslant -\overline{v}/\mu(v)$. □

Next, consider the modification of program (3-D) where we hold the optimal λ^* fixed and impose an additional constraint on q that depends on a parameter $\epsilon(k) \geqslant 0$:

$$\begin{aligned}
&\max_{\Xi:X(k)\to\mathbb{R},\, q:X(k)\to\mathbb{R}^N_+} \sum_{x\in X(k)} \rho(x)\Xi(x) + \sum_{v\in V}\mu(v)\lambda^*(v)\\
\text{s.t. } &\Xi(x) + \lambda^*(v) \leqslant v\cdot\nabla^- q(x)\ \forall v,x;\\
&\nabla^-_i q(x) = \begin{cases} kq_i(x) & \text{if } x_i = 0;\\ k(q_i(x) - q_i(x_i - 1/k, x_{-i})) & \text{if } 0 < x_i < k;\\ q_i(x) - q_i(x_i - 1/k, x_{-i}) & \text{if } x_i = k,\end{cases} \quad \forall i,x;\\
&\sum_{i=1}^N q_i(x) \leqslant 1\ \forall x;\\
&q_i(x_i - 1/k, x_{-i}) - q_i(x) \leqslant \epsilon(k)\ \forall i,x \text{ such that } 0 < x_i < k.
\end{aligned} \tag{31}$$

The dual of (31) is:

$$\begin{aligned}
&\min_{\substack{\gamma:X(k)\to\mathbb{R}_+,\,\sigma:X(k)\times V\to\mathbb{R}_+,\\ \zeta:X(k)\to\mathbb{R}^N_+}} \sum_x \gamma(x) - \sum_{v,x}\lambda^*(v)(\sigma(v,x) - \mu(v)) + \sum_{i,x}\zeta_i(x)\epsilon(k)\\
\text{s.t. } &\gamma(x) \geqslant \begin{cases} \sum_{v\in V} v_i k[\sigma(x,v) - \sigma(x_i + 1/k, x_{-i}, v)] & \\ \qquad +\zeta_i(x) - \zeta_i(x_i + 1/k, x_{-i}) & \text{if } x_i < k - 1/k;\\ \sum_{v\in V} v_i[k\sigma(x,v) - \sigma(x_i + 1/k, x_{-i}, v)] & \\ \qquad +\zeta_i(x) & \text{if } x_i = k - 1/k;\\ \sum_{v\in V} v_i\sigma(x,v) & \text{if } x_i = k,\end{cases}\\
&\quad \forall i,x;\\
&\zeta_i(0, x_{-i}) = 0\ \forall i, x_{-i};\\
&\sum_{v\in V}\sigma(x,v) = \rho(x)\ \forall x,
\end{aligned} \tag{32}$$

where $\zeta_i(x)$ is the multiplier on the constraint $q_i(x_i - 1/k, x_{-i}) - q_i(x) \leqslant \epsilon(k)$.

Lemma 9. *Suppose $\epsilon(k) > \frac{C}{k}$ where C is any constant bigger than $\frac{2\overline{v}/\mu(v)}{v''_i - v'_i}$ for all i, $v \in V$, and $v'_i, v''_i \in V_i$ such that $v'_i \neq v''_i$. Let $(\gamma^*, \sigma^*, \zeta^*)$ be an optimal solution to (32). Let $\overline{v}_i = \max V_i$ and $\underline{v}_i = \min V_i$. Suppose that $\mu(v) > 0$ for all $v \in V$. Then $\zeta^*_i(x) = 0$ for all i and $x \in X(k)$.*

Proof of Lemma 9. For the sake of contradiction, let x be a signal profile with the lowest Σx such that $\zeta_i^*(x) > 0$ for some i. Notice that by construction, $0 < x_i < k$. Let $w^*(x)$ be the interim expected values at x under σ^*:

$$w^*(x) = \frac{1}{\rho(x)} \sum_{v \in V} v\, \sigma^*(x, v).$$

Case 1: $w_i^*(x) < \overline{v}_i$.

In this case, there must exist a v such that $v_i < \overline{v}_i$ and $\sigma^*(x, v) > 0$. Fix such a v, and define

$$\overline{\sigma}(x', v') = \begin{cases} (1-\tau)\sigma^*(x', v) & \text{if } x' = x, v' = v; \\ \tau\sigma^*(x', v) + \sigma^*(x', v') & \text{if } x' = x, v' = (\overline{v}_i, v_{-i}); \\ \sigma^*(x', v') & \text{otherwise.} \end{cases}$$

Choose $\tau > 0$ such that

$$0 < \sum_{v' \in V} v_i' \overline{\sigma}(x, v')k - \sum_{v' \in V} v_i' \sigma^*(x, v')k = (\overline{v}_i - v_i)\tau\sigma^*(x, v)k \leqslant \zeta_i^*(x).$$

Set

$$\overline{\zeta}_i(x') = \begin{cases} \zeta_i^*(x) - (\overline{v}_i - v_i)\tau\sigma^*(x, v)k & \text{if } x' = x; \\ \zeta_i^*(x') & \text{otherwise,} \end{cases} \qquad \overline{\gamma}(x') = \gamma^*(x').$$

By construction, $(\overline{\gamma}, \overline{\sigma}, \overline{\zeta})$ is feasible for (32). Notice that

$$\sum_{v' \in V} \lambda^*(v')\overline{\sigma}(x, v') - \sum_{v' \in V} \lambda^*(v')\sigma^*(x, v') = (\lambda^*(\overline{v}_i, v_{-i}) - \lambda^*(v))\tau\sigma^*(x, v).$$

Therefore, the difference between the objectives of $(\gamma^*, \sigma^*, \zeta^*)$ and $(\overline{\gamma}, \overline{\sigma}, \overline{\zeta})$ in (32) is:

$$\left(\sum_{x'} \gamma^*(x') - \sum_{v', x'} \lambda^*(v')\sigma^*(v', x') + \sum_{i, x'} \zeta_i^*(x')\epsilon(k)\right) - \left(\sum_{x'} \overline{\gamma}(x') - \sum_{v', x'} \lambda^*(v')\overline{\sigma}(v', x') + \sum_{i, x'} \overline{\zeta}_i(x')\epsilon(k)\right)$$
$$= \epsilon(k)(\overline{v}_i - v_i)\tau\sigma^*(x, v)k + (\lambda^*(\overline{v}_i, v_{-i}) - \lambda^*(v))\tau\sigma^*(x, v) > 0,$$

since $\epsilon(k)k > (\lambda^*(\overline{v}_i, v_{-i}) - \lambda^*(v))/(\overline{v}_i - v_i)$ by Lemma 8. This is a contradiction.

Case 2: $w_i^*(x) = \overline{v}_i$ and $w_i^*(x_i - 1/k, x_{-i}) > \underline{v}_i$.

In this case, there must exist a v such that $v_i > \underline{v}_i$ and $\sigma^*(x_i - 1/k, x_{-i}, v) > 0$. Fix such a v, and define

$$\overline{\sigma}(x', v') = \begin{cases} (1-\tau)\sigma^*(x', v) & \text{if } x' = (x_i - 1/k, x_{-i}), v' = v; \\ \tau\sigma^*(x', v) + \sigma^*(x', v') & \text{if } x' = (x_i - 1/k, x_{-i}), v' = (\underline{v}_i, v_{-i}); \\ \sigma^*(x', v') & \text{otherwise.} \end{cases}$$

Choose $\tau > 0$ such that

$$0 < \sum_{v' \in V} v_i' \sigma^*(x_i - 1/k, x_{-i}, v')k - \sum_{v' \in V} v_i' \overline{\sigma}(x_i - 1/k, x_{-i}, v')k = (v_i - \underline{v}_i)\tau\sigma^*(x_i - 1/k, x_{-i}, v)k \leqslant \zeta_i^*(x).$$

Set

$$\overline{\zeta}_i(x') = \begin{cases} \zeta_i^*(x) - (v_i - \underline{v}_i)\tau\sigma^*(x_i - 1/k, x_{-i}, v)k & \text{if } x' = x; \\ \zeta_i^*(x') & \text{otherwise,} \end{cases} \qquad \overline{\gamma}(x') = \gamma^*(x').$$

As in Case 1, $(\overline{\gamma}, \overline{\sigma}, \overline{\zeta})$ is feasible for (32) and has a strictly lower objective than $(\gamma^*, \sigma^*, w^*, \zeta^*)$, a contradiction.

Case 3: $w_i^*(x) = \overline{v}_i$ and $w_i^*(x_i - 1/k, x_{-i}) = \underline{v}_i$.

The virtual value at $(x_i - 1/k, x_{-i})$ is $\underline{v}_i - (k-1)(\overline{v}_i - \underline{v}_i) < 0$ when k is sufficiently large. Since $\gamma^*(x_i - 1/k, x_{-i}) \geqslant 0$ and $\zeta_i^*(x_i - 1/k, x_{-i}) = 0$, we must have

$$\gamma^*(x_i - 1/k, x_{-i}) > \sum_{v' \in V} v_i' k[\sigma^*(x_i - 1/k, x_{-i}, v') - \sigma^*(x, v')] + \zeta_i^*(x_i - 1/k, x_{-i}) - \zeta_i^*(x),$$

so we can decrease $\zeta_i^*(x)$ to lower strictly the objective of $(\gamma^*, \sigma^*, \zeta^*)$, a contradiction. □

Lemma 10. *Suppose $\mu(v) > 0$ for every $v \in V$. Let $\epsilon(k) > \frac{C}{k}$ where C is any constant bigger than $\frac{2\overline{v}/\mu(v)}{v_i'' - v_i'}$ for all i, $v \in V$, and $v_i', v_i'' \in V_i$ such that $v_i' \neq v_i''$. Then there exists an optimal solution (λ^*, Ξ^*, q^*) of (3-D) that satisfies*

$$q_i^*(x_i - 1/k, x_{-i}) \leqslant q_i^*(x) + \epsilon(k) \tag{33}$$

for every i and every x such that $0 < x_i < k$.

Proof of Lemma 10. Lemma 9 implies that program (31) has the same value even if we drop the constraints (33). This implies the result. □

Proof of Lemma 6. Let $\epsilon(k) = \frac{C+\epsilon}{k}$ where C is given in the statement of Lemma 10 and $\epsilon > 0$ is arbitrary. By Lemma 10, let (λ^*, Ξ^*, q^*) be an optimal solution of (3-D) that satisfies $q_i^*(x_i - 1/k, x_{-i}) \leqslant q_i^*(x) + \epsilon(k)$ for every i and every x such that $0 < x_i < k$.

Define

$$\begin{aligned} q_i'(x) &= \begin{cases} \frac{q_i^*(x_i - 1/k, x_{-i})}{1 + N\epsilon(k)} & \text{if } 0 < x_i < k; \\ 0 & \text{if } x_i = 0 \text{ or } x_i = k; \end{cases} \\ \lambda'(v) &= \frac{k-1}{k(1 + N\epsilon(k))}\lambda^*(v) \quad \forall v \in V; \\ \Xi'(x) &= \begin{cases} \frac{k-1}{k(1+N\epsilon(k))}\Xi^*(x) & \text{if } x \notin \partial X(k); \\ -(k-1)N\overline{v} - \max_{v \in V} \lambda'(v) & \text{if } x \in \partial X(k), \end{cases} \end{aligned} \tag{34}$$

where $\partial X(k) = \{x \in X(k) \mid x_i \geqslant k - 1/k \text{ for some } i\}$.

We claim that (λ', Ξ', q') is feasible for the program (4′): First, the constraint (4′a) holds for $x \in \partial X(k)$ because

$$\Xi'(x) = -(k-1)N\overline{v} - \max_{v \in V} \lambda'(v) \leqslant v \cdot \nabla^+ q'(x) - \lambda'(v)$$

for all v; (4′a) also holds for $x \notin \partial X(k)$ because $\nabla^+ q'(x) = \frac{k-1}{k(1+N\epsilon(k))} \nabla^- q^*(x)$, $\Xi'(x) = \frac{k-1}{k(1+N\epsilon(k))} \Xi^*(x)$, $\lambda'(v) = \frac{k-1}{k(1+N\epsilon(k))} \lambda^*(v)$, and $\Xi^*(x) + \lambda^*(v) \leqslant v \cdot \nabla^- q^*(x)$. Also, the feasibility constraint (4′b) is satisfied, as

$$\sum_{i=1}^{N} q_i'(x) = \sum_{i=1}^{N} \frac{q_i^*(x_i - 1/k, x_{-i})}{1 + N\epsilon(k)} \mathbb{I}_{0 < x_i < k} \leqslant \sum_{i=1}^{N} \frac{q_i^*(x) + \epsilon(k)}{1 + N\epsilon(k)} \leqslant 1.$$

Finally, the difference in objectives of (3-D) under (λ^*, Ξ^*, q^*) (which is equal to $\overline{\Pi}^{\text{MIN-2MAX}}(k)$) and of (4′) under (λ', Ξ', q') is

$$\begin{aligned}
&\sum_{x \in X(k)} \rho(x)(\Xi^*(x) - \Xi'(x)) + \sum_{v \in V} \mu(v)(\lambda^*(v) - \lambda'(v)) \\
&= \sum_{x \in X(k)} \rho(x) \left(1 - \frac{k-1}{k(1+N\epsilon(k))}\right) \Xi^*(x) + \sum_{x \in \partial X(k)} \rho(x) \left(\frac{k-1}{k(1+N\epsilon(k))} \Xi^*(x) - \Xi'(x)\right) \\
&\quad + \sum_{v \in V} \mu(v) \left(1 - \frac{k-1}{k(1+N\epsilon(k))}\right) \lambda^*(v) \\
&= \left(1 - \frac{k-1}{k(1+N\epsilon(k))}\right) \overline{\Pi}^{\text{MIN-2MAX}}(k) \\
&\quad + \sum_{x \in \partial X(k)} \rho(x) \left(\frac{k-1}{k(1+N\epsilon(k))} \Xi^*(x) + (k-1)N\overline{v} + \max_{v \in V} \frac{k-1}{k(1+N\epsilon(k))} \lambda^*(v)\right) \\
&\leqslant \left(1 - \frac{k-1}{k(1+N\epsilon(k))}\right) \overline{\Pi}^{\text{MIN-2MAX}}(k) + N(1-1/k)^{k^2-1} \left(\frac{k-1}{k(1+N\epsilon(k))} kN\overline{v} + (k-1)N\overline{v}\right),
\end{aligned}$$

where in the last line we use the fact that $\rho(\partial X(k)) \leqslant N(1-1/k)^{k^2-1}$ and $\Xi^*(x) + \lambda^*(v) \leqslant v \cdot \nabla^- q^*(x) \leqslant kN\overline{v}$. The last line of the display equation vanishes as $k \to \infty$ because $\epsilon(k) \to 0$ and $\overline{\Pi}^{\text{MIN-2MAX}}(k) \leqslant \overline{v}$ for all k (since $\sigma(x, v) = \rho(x)\mu(v)$ is feasible for the program (3)). This implies the result, since $\underline{\Pi}^{\text{MAX-2MIN}}(k)$ is equal to the value of the program (4′), which is weakly larger than the objective obtained by (λ', Ξ', q'). □

B Online Appendix

B.1 More Examples and Extensions

B.1.1 Perfectly correlated values

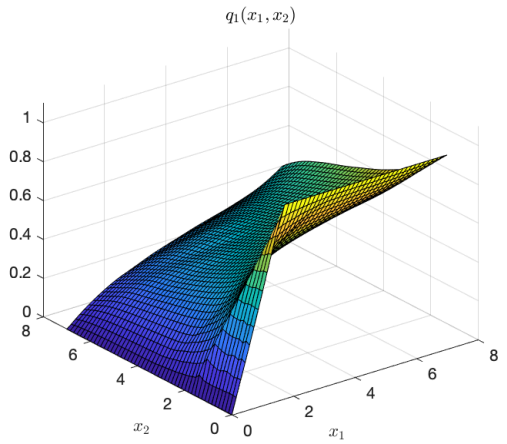
We now present an example with $N = 2$ and $v_1 = v_2 + c$ for a constant c, i.e., values are perfectly positively correlated. Bidder 2's value v_2 is uniformly distributed on an evenly spaced grid of 10 values between 0 and 1.

Note that this example does not satisfy the full-support hypothesis, so that the asymptotic equivalence of (1) and (2) is not implied by Theorem 1. When values are pure common ($c = 0$), however, the equivalence of these programs is a result of the constructive argument in Theorems 3 and 4 of Brooks and Du (2020). And the first part of Theorem 1, that the local relaxations provide profit guarantees, does not depend on μ having full support. In fact, we conjecture that Theorem 1 remain true in their entirety even without full support.

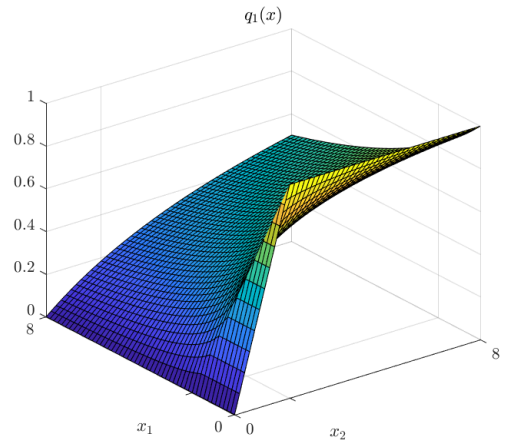
We now proceed with the discussion of simulations. In the case of pure common value ($c = 0$), Brooks and Du (2020) presented max-2min mechanisms and min-2max information structures in the limit when the action/signal space is all of $\mathbb{R}_+$. The mechanism has the form of a "proportional auction," in which the aggregate allocation and aggregate transfer only depend on the aggregate action, and individual allocations and transfers are proportional to actions. For this example, the aggregate allocation has the form $Q(\Sigma x) = \min\{1, \alpha\Sigma x\}$ for a constant α. Thus, each bidder i's allocation on the low rationing region is a simple linear function of their action: $q_i(x) = \alpha x_i$. (This appears to be a general feature of aggregate allocations for mechanisms in which the good is rationed for low aggregate actions.)

The first row of Figure 2 shows the approximate max-2min mechanism as computed by solving (4) with $k = 7$ (so that each bidder has 50 actions). This mechanism guarantees profit of at least 0.2564, or 51% of the expected value. The proportional auction (for which the message space is all of $\mathbb{R}_+$) is depicted in the second row for comparison. The approximate max-2min allocation bears a close resemblance to the proportional rule, including in the behavior of the aggregate allocation. Indeed, the solution in Brooks and Du (2020) was in part motivated by looking at simulations of this form. The approximate max-2min transfer does not suggest the proportional form. As we will discuss in Section 5, even holding fixed a particular max-2min allocation, there may be many transfer rules which could complete a max-2min mechanism. Numerical simulations of (4) need not produce the most interesting or tractable solution. As a result, for our subsequent examples, we will focus on max-2min allocations, and revisit the question of max-2min transfers in Section 5.

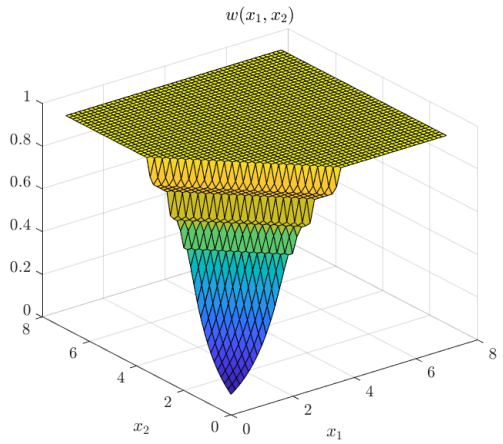
The bottom panel of Figure 2 shows the approximate min-2max information produced by (3). Profit in this information structure is at most 0.2856, so that the gap between $\underline{\Pi}^{\text{MAX}-2\text{MIN}}(k)$ and $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$ is approximately 5.8% of the expected value. Interestingly, the simulated min-2max information very nearly coincides with the theoretical solution with a continuum of signals: The interim expected value $w_1(x) = w_2(x)$ to be an increasing function of the aggregate signal. There is a cutoff, below which the interim expected value grows exponentially, and above which the interim expected value is equal to the ex post value. This structure gives rise to the discontinuities in the value function,



(a) Approximate max-2min allocation

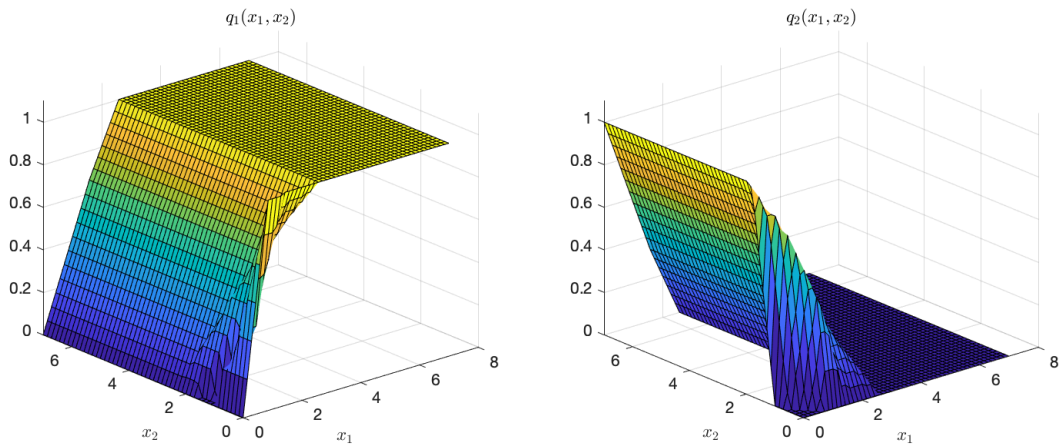


(b) Limit max-2min allocation

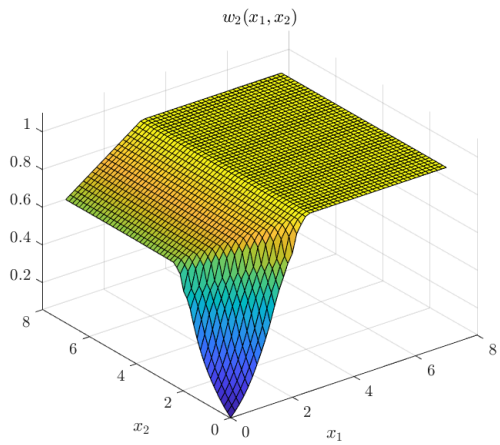


(c) Approximate min-2max information

Figure 2: Max-2min allocations and min-2max information with pure common values.



(a) Approximate max-2min allocations



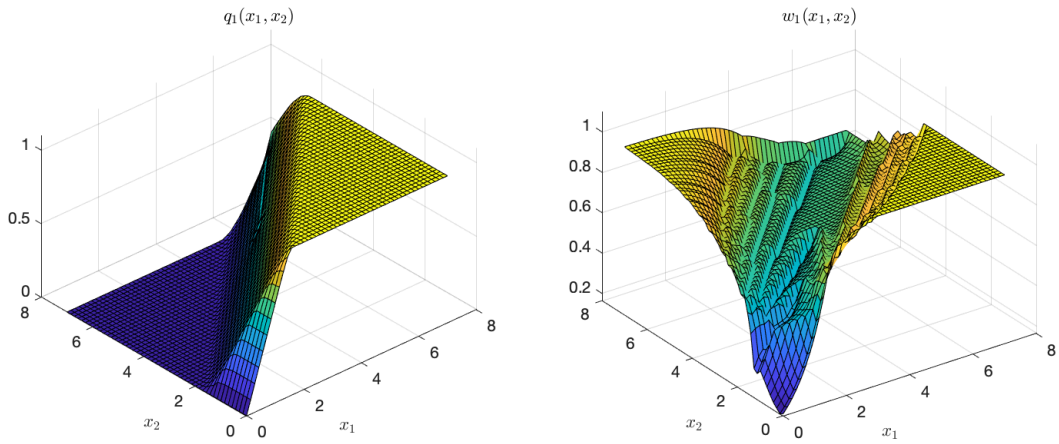
(b) Approximate min-2max interim value

Figure 3: Approximate max-2min mechanism and min-2max information with perfectly correlated asymmetric values ($c = 0.1$).

evident in Figure 2, which occur when the interim expected value jumps up to the next higher value in the grid with increments of 0.1.

Our next simulation has $c = 0.1$, so that there is common knowledge that bidder 1's value is higher than bidder 2's.[22] In this case, it is socially efficient to always allocate the good to buyer 1. In contrast, the max-2min mechanism takes into account the cost of incentivizing truthtelling, and sometimes allocates to bidder 2 so as to reduce information rents. The simulated allocation is depicted in the top row of Figure 3. Together with the transfer that solves (4), this mechanism guarantees profit at least 0.2977, while the

[22]Note that this model does not have pure common values and is not characterized by Brooks and Du (2020).



(a) Approximate max-2min allocation

(b) Approximate min-2max interim value

Figure 4: Approximate max-2min mechanism and min-2max information for independent values.

efficient surplus (if the good is always allocated to bidder 1) is now 0.6.[23] Remarkably, it appears that for signals in which the aggregate allocation is 1, the allocation only depends on bidder 1's signal. The approximate min-2max information structure has independent censored geometric signals, with the interim value function for v_2 depicted in the bottom row of Figure 3. (Bidder 1's interim expected value is simply $w_1(x) = w_2(x) + 0.1$.) Profit on this information structure is at most 0.2856. Thus, while this example does not satisfy the full support assumption, we see that the upper and lower bounds on profit are quite close. Note that bidder 1's allocation hits 1, and bidder 2's allocation hits 0, precisely on the region where the interim expected value maxes out, i.e., $w_1 = 1.1$ and $w_2 = 1$.

As c increases, the region where the allocation is interior shrinks. When c is sufficiently large, the optimal mechanism always allocates the good to bidder 1 at a price of 1.

B.1.2 Independent values

Our next example involves two bidders whose values are independently distributed on the same ten-point grid in $[0, 1]$. The simulated allocation and interim values for bidder 1 are depicted in Figure 4. The corresponding objects for bidder 2 are symmetric.

The allocation that solves (4) is in the left panel. The expected highest value in this discrete example is $0.68\overline{3}$, and the approximate max-2min mechanism guarantees profit of at least 0.2826, or approximately 41% of the efficient surplus.[24] We again see a region where the aggregate action is below a cutoff on which the good is rationed, and each bidder's

[23]Note that the profit guarantee rises by much less than the increase in the efficient surplus, because in order to realize that gain, it would be necessary to allocate the good to bidder 1, which in turn would necessitate granting bidder 1 a large information rent.

[24]Thus, while the profit guarantee is higher than with perfectly correlated values, it does not rise nearly as much as the expected value. The reason, of course, is that the bidders can obtain higher information rents when their values are independent.

allocation is linear in their action. A striking result is that on the high region where the good is always allocated, it appears that the allocation only depends on the difference in the bidders' signals, with a bidder's allocation being increasing in their action.

The interim expected value is on the right panel. Maximum profit on this information structure is at most 0.3170. While bidders' ex post values are independent, their interim expectations are highly correlated, with both bidders' interim expected values being higher when the absolute difference in their signals is large. Bidder i's interim value is highest when $x_i - x_j$ is above a threshold. The set of signals where bidder i's interim value is maximized roughly corresponds to the set of actions where their allocation hits 1.

B.1.3 Constraints on the allocation

The proof of Theorems 2 and 1 can be extended to a variety of auction design problems which are not formally subsumed in the model of Section 2. We now describe three such variations: Constraints on the allocation, multiple goods, and ambiguity about the value distribution.

Even if bidders have the same per-unit value for the good, they may demand different quantities. For example, if the auction is an IPO, then different bidders may have different capacities for the risk associated with owning equity in the firm. These different risk capacities may be related to public information about the bidders, such as the sizes of their portfolios.

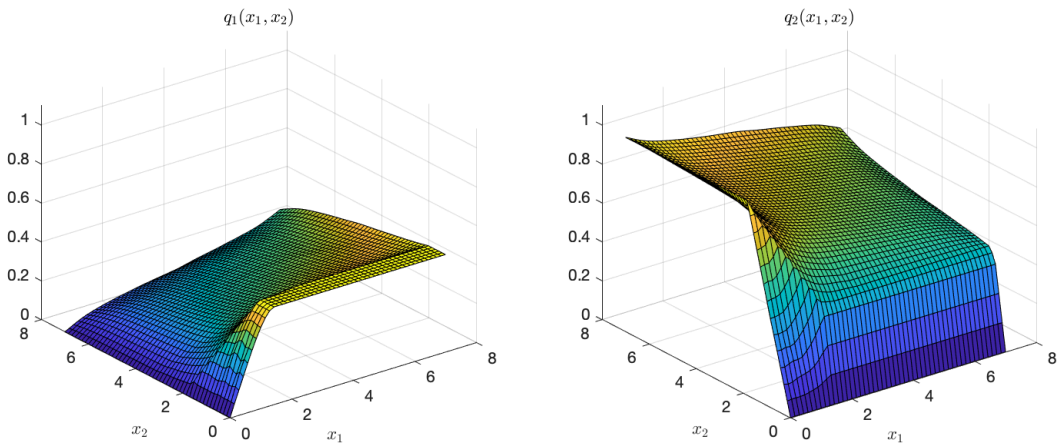
Let us suppose that it is public information that bidder i demands at most κ_i units of the good. A mechanism must now satisfy the additional restrictions $q_i(x) \leqslant \kappa_i$ for all i.

We claim that Theorems 1 and 2 can be generalized to models with such asymmetric demands, as we now explain. The program (3) is modified to the following:

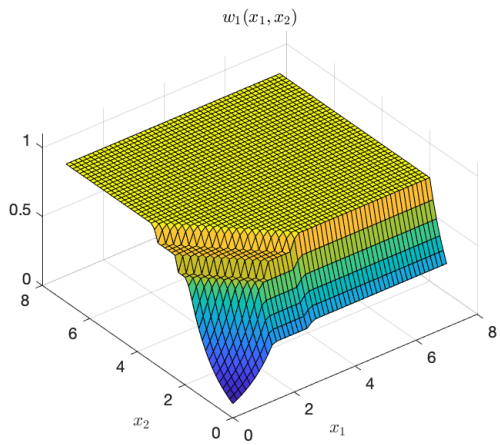
$$\begin{aligned}
&\min_{\gamma:X(k)\to\mathbb{R}_+,\eta_i:X(k)\to\mathbb{R}_+,\sigma:X(k)\times V\to\mathbb{R}_+,w:X(k)\to\mathbb{R}^N_+} \sum_{x\in X(k)} \left[\gamma(x) + \sum_i \kappa_i \eta_i(x)\right] \\
\text{s.t. } &\gamma(x) + \eta_i(x) \geqslant \rho(x)\left[w_i(x) - \nabla_i^+ w(x)\right] \ \forall x; \\
&\sum_{v\in V} \sigma(x,v) = \rho(x) \ \forall x; \\
&\sum_{x\in X(k)} \sigma(x,v) = \mu(v) \ \forall v \\
&w(x) = \frac{1}{\rho(x)} \sum_{v\in V} v\sigma(x,v) \ \forall x.
\end{aligned} \tag{35}$$

while (4) becomes

$$\begin{aligned}
&\max_{q:X(k)\to A, t:X(k)\to\mathbb{R}^N, \lambda:V\to\mathbb{R}} \sum_{v\in V} \mu(v)\lambda(v) \\
\text{s.t. } &\lambda(v) \leqslant \Sigma t(x) + v\cdot\nabla^+ q(x) - \nabla^+ \cdot t(x) \ \forall (v,x); \\
&\Sigma q(x) \leqslant 1 \ \forall x; \\
&q_i(x) \leqslant \kappa_i \ \forall i, x; \\
&t_i(0, x_{-i}) = 0 \ \forall i, x_{-i}.
\end{aligned}$$



(a) Approximate max-2min allocations



(b) Approximate min-2max interim value

Figure 5: Approximate max-2min mechanism and min-2max information with pure common values but bidder 1 demands only 0.5 units.

The proof that these programs bound (1) and (2) follows closely the proofs of Lemmas 2 and 3, with the additional demand constraints. The variables η_i in (35) are the multipliers on the demand constraints in the inner minimization program of (1) and are introduced when we take a dual as in the proof of Lemma 2. The proof of Lemma 3 is essentially unchanged.

In addition, the arguments for bounded λ in Lemma 10 and the shifting argument of Lemma 6 proceed essentially as before. A subtle feature of the proof of Lemma 10 is that in the perturbation which "drives out" the multipliers on the constraints (33), we change the value distribution but do not change γ. When we add demand constraints, the perturbation proceeds as before, and now both γ and η_i are unchanged. For Lemma 6, a key step is the transformation of an optimal solution (λ^*, Ξ^*, q^*) to (3-D) into a feasible solution $(\overline{q}, \overline{\lambda}, \overline{\Xi})$ for the dual of (4), defined in (34) which has approximately the same value. Critically, we

have defined $\overline{q}$ to be either 0 or so that

$$\overline{q}_i(x) \leqslant q_i^*(x_i - 1/k, x_{-i}) \leqslant \kappa_i,$$

so that $\overline{q}$ also satisfies individual demand constraints. The rest of the proof goes through as before.

Figure 5 depicts the approximate max-2min mechanism and min-2max information for the pure common value example of Section 4, except that bidder 1 is only willing to buy 0.5 units of the good. In addition to the usual low linear rationing region, we see that there is a rectangular region where bidder 1's action is relatively high and bidder 2's action is relatively low on which bidder 1's allocation maxes out at 0.5. On this region, the good is still rationed, and bidder 2's allocation only depends on their signal. In the information structure, the interim expected value only depends on bidder 2's signal on the region where bidder 1's allocation has maxed out.

Note that the particular form for the feasibility constraint does not play a significant role in the argument, and we could easily generalize to other constraints on the allocation, e.g., a cap on the share of the good allocated to a subset of the bidders. The critical feature is that the constraint is "downward closed," meaning that if all bidders' allocations decrease, the constraint will still be satisfied.

B.1.4 Simultaneous auction of multiple goods

Suppose that there are L goods for sale, indexed by $l = 1, \ldots, L$, and each bidder demands a single unit of each of the goods. Let $v_{l,i}$ denote bidder i's value for good l. The primitive is a prior distribution over buyers' values for all goods. An information structure now consists of sets of signals and a joint distribution over signals and bidders' values for all goods. A mechanism now specifies sets of actions and allocations for each bidder and good.[25]

The program (3) is generalized as follows:

$$\begin{aligned}
&\min_{\gamma_l : X(k) \to \mathbb{R}_+,\, \sigma : X(k) \times V^L \to \mathbb{R}_+,\, w_l : X(k) \to \mathbb{R}_+^N} \sum_{l=1}^{L} \sum_{x \in X(k)} \gamma_l(x) \\
\text{s.t. } & \gamma_l(x) \geqslant \rho(x) \left[w_{l,i}(x) - \nabla_i^+ w_l(x) \right] \ \forall l, i, x; \\
& \sum_{v \in V} \sigma(x, v) = \rho(x) \ \forall x; \\
& \sum_{x \in X(k)} \sigma(x, v) = \mu(v) \ \forall v; \\
& w_l(x) = \frac{1}{\rho(x)} \sum_{v \in V} v_l \sigma(x, v) \ \forall l, x,
\end{aligned}$$

[25]The demand constraint example of the previous section could also be modeled with multiple goods, some of which are only assigned positive value by some bidders. Nonetheless, there is independent value to the extension with demand constraints, as it illustrates how more general feasibility constraints could be added to the model, including constraints which cannot be easily mapped into a multiple goods model.

where now $w(x)$ is a matrix that specifies an interim expected value $w_{l,i}(x)$ for each good l and bidder i. The program (4) becomes

$$\max_{q_l:X(k)\to\mathbb{R}^N_+,\, t_l:X(k)\to\mathbb{R}^N,\, \lambda:V^L\to\mathbb{R}} \sum_{v\in V^L} \mu(v)\lambda(v)$$
$$\text{s.t. } \lambda(v) \leqslant \sum_{l=1}^{L} \left(\Sigma t_l(x) + v_l \cdot \nabla^+ q_l(x) - \nabla^+ \cdot t_l(x)\right) \ \forall (v,x);$$
$$\Sigma q_l(x) \leqslant 1 \ \forall l, x;$$
$$t_{l,i}(0, x_{-i}) = 0 \ \forall l, i, x_{-i}.$$

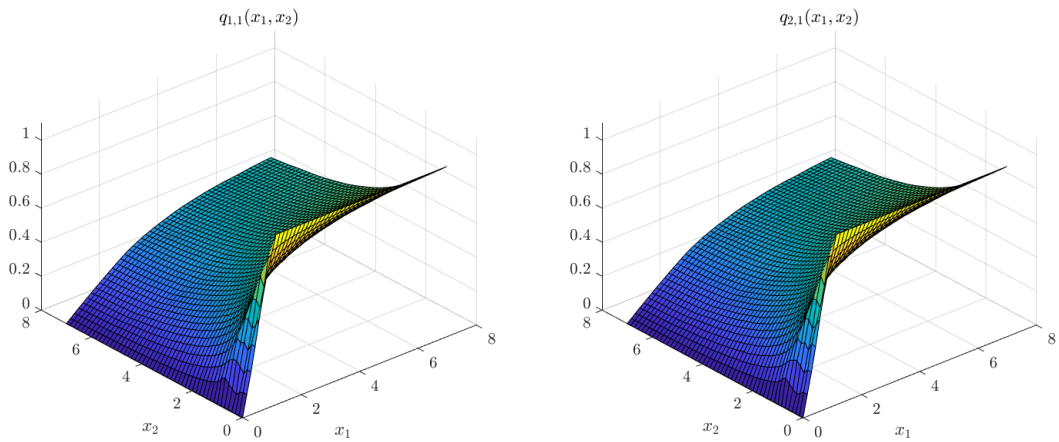
As with asymmetric demands, the proofs of Lemmas 2 and 3 proceed as before, by dropping non-local obedience and incentive constraints and fixing the multipliers on local incentives. Lemma 10 is also generalized, by showing that the constraints

$$q_{l,i}(x_i - 1/k, x_{-i}) \leqslant q_{l,i}(x) + \epsilon(k) \tag{36}$$

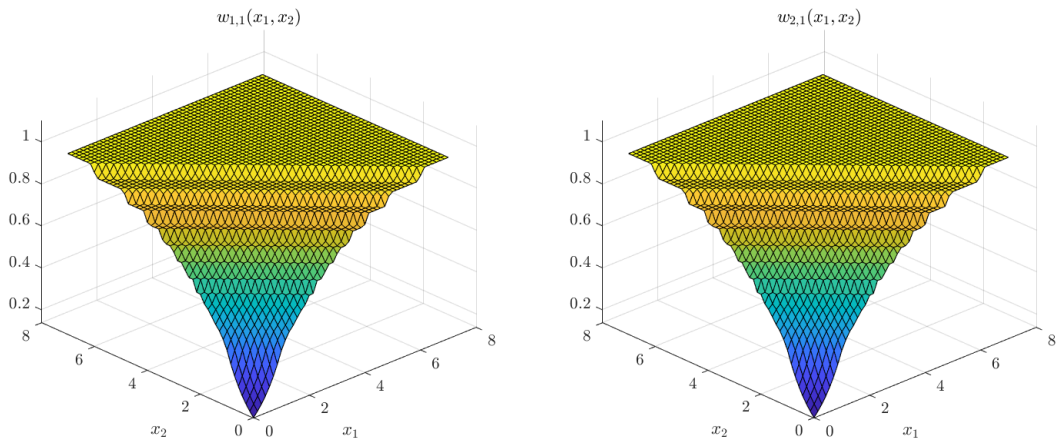
are redundant if $\epsilon(k) = C/k$ for C sufficiently large. The proof is as before, via a perturbation of the value distribution that drives out multipliers on (36). The proof of Lemma 6 is also generalized: when k is large, there is an optimal solution (λ^*, Ξ^*, q^*) that satisfies the no-downward-jump constraints (36) for each l and i. We can define a new solution $\overline{q}_{l,i}$ exactly as in (34), which has approximately the same value.

We illustrate this extension with a two-bidder two-good example. First assume bidders have pure common values for each good, so $v_{l,1} = v_{l,2}$ almost surely for each $l = 1, 2$. The common values are independently distributed across goods. The lower and upper bounds on Π^* are 0.5942 and 0.6542, respectively. The optimal mechanism and information are depicted in Figure 6. (Bidder 1's allocations are depicted, with bidder 2's allocations being symmetric.) The simulation clearly indicates that the allocation and interim expected values for the two goods are *exactly the same*. Thus, the two-good pure common value model reduces to a single-good pure common value model, in which the value for the single good is the sum of the values of the two goods. This is in spite of the fact that the underlying values are independent across the two goods. Why should this be the case? Clearly the seller can treat the two goods as one, and only offer them bundled together, in which case all that matters are bidders' beliefs about the value of the bundle. At the same time, it is possible that the information structure only gives bidders information about the value of the bundle, as in the simulation. In this case, the symmetry of the underlying value distribution implies that $w_{1,i}(x) = w_{2,i}(x)$ for all x, i.e., bidders always assign the same interim expected value to both goods. As a result, the seller can do no better than the optimal profit guarantee when the goods are bundled.

Indeed, we conjecture that the limit analysis of Brooks and Du (2020) can be generalized to formally show that proportional auctions for the grand bundle are max-2min mechanisms when there are multiple goods with pure common values and the distribution of the goods' values is exchangeable. More broadly, let us say that a value distribution is *exchangeable across goods* if all $\mu(v) = \mu(v')$ for all v and v', where v'_i is a permutation of v_i for all i. We conjecture that if values are exchangeable across goods, then the multi-good problem



(a) Approximate max-2min allocations



(b) Approximate min-2max interim value

Figure 6: Approximate max-2min mechanism and min-2max information with pure common values and two goods.

reduces to a single-good problem in which bidders only learn about their value for the grand bundle, and the seller only offers the grand bundle for sale.

If, however, values are not exchangeable across goods, then the multiple-good problem need not reduce to an auction for the grand bundle, as the following example shows. Let us now suppose that bidder 2's values $v_{l,2}$ are distributed as before, uniform on each good l and independent across goods; bidder 1 has the same value for good 2 as bidder 1 but assigns more value to good 1 than bidder 2: $v_{2,1} = v_{2,2}$ and $v_{1,1} = v_{1,2} + 1$. The approximate max-2min allocations are depicted in Figure 7. As we can see, bidders receive each good with different probabilities. As we would expect, good 1 is mostly allocated to bidder 1, since their value for that good is much higher. Interestingly, bidder 1 also tends to get more shares of good 2 than bidder 2, even though the two bidders have the same value for good 2, because of the endogenous bundling of the two goods in the max-2min mechanism.

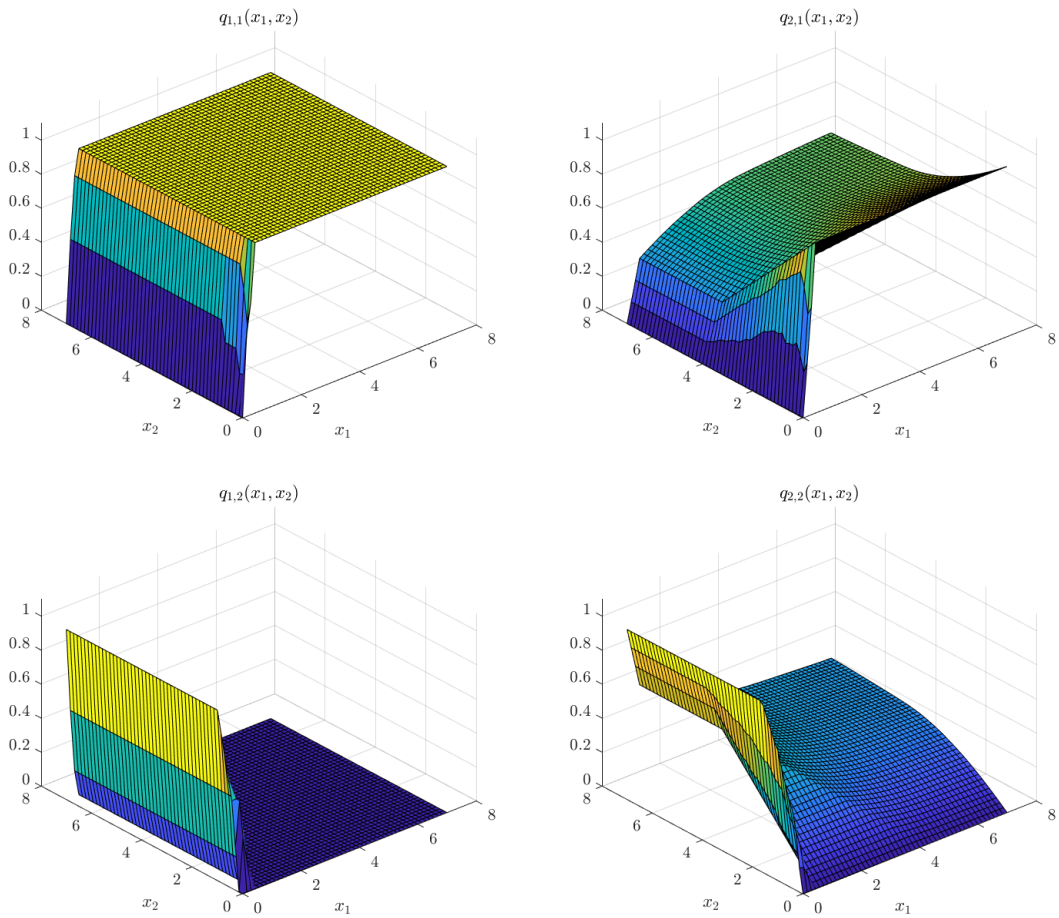


Figure 7: Max-2min allocation with two goods with non-exchangeable values.

B.1.5 Ambiguous correlation between values

Our last two extensions regards the constraint on the value distribution. We have assumed heretofore that the seller knows the value distribution exactly, while at the same time taking a worst case over bidders' information and the equilibrium strategies. There is a clear tension here. Our last two extensions incorporate ambiguity with regard to the value distribution.

First, suppose that the seller knows that each bidder i's value is distributed according to $\mu_i \in \Delta(V)$, but the seller does not know the joint distribution of values. Thus, an information structure (S, σ) need only satisfy

$$\sum_{s, v_{-i}} \sigma(s, v_i, v_{-i}) = \mu_i(v_i) \ \forall i, v_i.$$

These constraints replace the marginal constraint in the program (3). The analogue of program (4) is now:

$$\max_{q:X(k)\to\mathbb{R}^N_+, t:X(k)\to\mathbb{R}^N, \lambda:V\to\mathbb{R}} \sum_{i=1}^{N}\sum_{v\in V}\mu_i(v_i)\lambda_i(v_i)$$
$$\text{s.t. } \sum_{i=1}^{N}\lambda_i(v_i) \leqslant \Sigma t(x) + v\cdot\nabla^+ q(x) - \nabla^+\cdot t(x)\ \forall(v,x);$$
$$\Sigma q(x) \leqslant 1\ \forall x;$$
$$t_i(0,x_{-i}) = 0\ \forall i, x_{-i}.$$

All of the previous steps in our argument go through as before, where we replace $\lambda(v)$ with $\sum_i \lambda_i(v_i)$, with one exception. In the proof of Lemma 10, we invoked a full-support hypothesis on μ to prove that optimal λ are bounded. But now, λ will be bounded if each component λ_i is bounded, and the same argument for boundedness of λ_i goes through as long as μ_i has full support on V_i. In particular, we can always normalize the λ_i so that

$$\sum_{v_i\in V_i}\mu_i(v_i)\lambda_i(v_i) = \frac{\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)}{N}$$

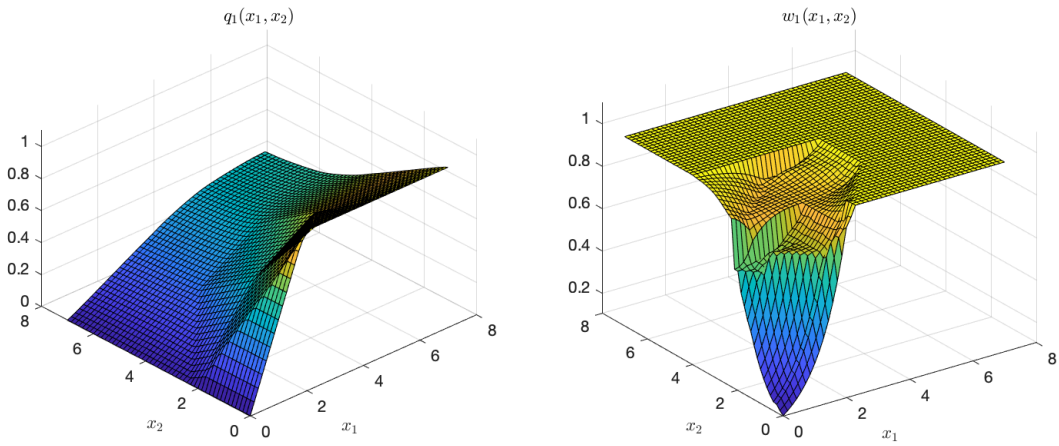
for all i. Under this normalization, λ_i must be bounded above, since otherwise, in the version of (3-D) with fixed λ_i, there are feasible solutions that place all of the mass on values with $\lambda_i(v_i)$ going to infinity, which would contradict the hypothesis that the value of the program is $\overline{\Pi}^{\text{MIN}-2\text{MAX}}(k)$. And since the optimal value of (3-D) is bounded below, λ_i must be bounded below as well. Given this result, the proof of Lemma 6 goes through unchanged.

We illustrate with two bidders whose values are uniformly distributed on the ten point grid in $[0,1]$. The optimal allocation and interim value are illustrated in Figure 8. The mechanism guarantees profit at least 0.2537 and maximum profit on the information structure is at most 0.2834. Note that both of these numbers are lower than the corresponding figures for pure common uniform values (0.2564 and 0.2856) and independent uniform values (0.2826 and 0.3170), as they should be, since perfect correlation and independence are both feasible joint distributions for the present problem.

B.1.6 A penalty-based model of ambiguous value distribution

Going a step further, we can dispense with marginal constraints on values altogether, and instead represent the seller's ambiguity about the value distribution with "cost" for the value distribution. This is in the spirit of the multiplier preferences Hansen and Sargent (2001) and the variational preferences of Maccheroni, Marinacci, and Rustichini (2006), where a penalty function on beliefs is used to discipline a worst-case analysis.

We now argue that Theorems 2 and 1 can be generalized to such a model of ambiguity aversion. In fact, we have already analyzed this model in the proof of Lemma 10, where the penalty function arose endogenously as part of the solution of the program (4). Fix a



(a) Approximate max-2min allocation (b) Approximate min-2max interim value

Figure 8: Approximate max-2min mechanism and min-2max information with uniformly distributed values and unknown correlation.

penalty $\lambda : V \to \mathbb{R}$. The programs (1) and (2) remain the same, except that the domain of information structures is now the set of finite information structures with arbitrary marginal on V, and the objective is now

$$\Pi(\mathcal{M}, \mathcal{I}, \beta) - \sum_{v \in V} \lambda(v) \sum_{s \in S} \sigma(s, v).$$

The program (3) is modified by dropping the marginal constraints on V and changing the objective to

$$\sum_{x \in X(k)} \gamma(x) - \sum_{v \in V} \lambda(v) \sum_{x \in X(k)} \sigma(x, v).$$

The only change to (4) is that λ is fixed and is no longer a variable over which we optimize.

Lemmas 2 and 3 go through as before, with the penalty replacing the marginal constraint on values. In the proof of Lemma 10, boundedness of the optimal λ holds by assumption, and no full support condition on the (endogenous) distribution of values is needed. The only significant change is in the proof of Lemma 6, since the new solution as defined in (34) is no longer feasible (since we cannot change the exogenously given λ). We can instead define a solution for the analogue of (4′):

$$\overline{q}_i(x) = \begin{cases} \frac{q_i^*(x_i - 1/k, x_{-i})}{1 + N\epsilon(k)} & \text{if } 0 < x_i < k; \\ 0 & \text{if } x_i = 0 \text{ or } x_i = k; \end{cases}$$

$$\overline{\Xi}(x) = \begin{cases} \frac{k-1}{k(1+N\epsilon(k))} \Xi^*(x) & \text{if } x \notin \partial X(k); \\ -(k-1)N\overline{v} - \max_{v \in V} \lambda(v) & \text{if } x \in \partial X(k), \end{cases}$$

$$- \left(1 - \frac{k-1}{k(1 + N\epsilon(k))}\right) \max_{v \in V} \lambda(v)$$

It is straightforward to verify that this solution is feasible for (4′) with fixed λ and has approximately the same value as (34) when k is large.

B.2 Construction of transfers for $N > 2$

In this section we give a general construction of participation secure transfers via "balanced divisions" of an aggregate excess growth. Let us omit k and write $X_i = X_i(k)$. For any subset $N' \subseteq \{1, 2, \ldots, N\}$ of bidders, let $X_{N'} = \prod_{i \in N'} X_i$ and $\rho(x) = \prod_{i \in N'} \rho_i(x_i)$ for $x \in X_{N'}$.

Let Z be the set of non-repeating sequences in $\{1, \ldots, N\}$ of length less than or equal to N. We also define $Z(i) \subseteq Z$ to be the set of sequences of length less than or equal to $N-1$ which does not contain i. Given $z \in Z$, we let $N(z)$ be the set of players not in z. And for $z \in Z(i)$, we let (z, i) be the sequence that catenates z and i.

Fix an aggregate excess growth Ξ that is balanced. We say that a collection $\{\Xi_z\}_{z \in Z}$ is a *balanced division (of Ξ)* if the following conditions are satisfied:

1. $\Xi_\varnothing = \Xi$.
2. For all i, $z \in Z(i)$, $\Xi_{(z,i)} : X_i \times X_{N(z,i)} \to \mathbb{R}$.
3. For all i, $z \in Z(i)$, $\sum_{x \in X_i \times X_{N(z,i)}} \Xi_{(z,i)}(x)\rho(x) = 0$.
4. For all i, $z \in Z(i)$, and $x_{N(z,i)} \in X_{N(z,i)}$,
$$\sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)})\rho_i(x_i) = \sum_{j \in N(z,i)} \Xi_{(z,i,j)}(x_{N(z,i)}).$$
5. For all $x \in X_{N(\varnothing)} = \prod_{j=1}^{N} X_j$,
$$\Xi_\varnothing(x) = \sum_{j \in N(\varnothing)} \Xi_j(x).$$

We interpret $\Xi_{(z,i)}$ as bidder i's excess growth (before adjustment) when the bidders in z are ahead of i in a queue. Condition 2 says that the excess growth $\Xi_{(z,i)}$ depends x_i and $x_{N(z,i)}$, but not on the actions of the bidders in z. Condition 3 says that $\Xi_{(z,i)}$ is balanced. Condition 4 says that the adjustment one makes to i's excess growth (see equation (37)) is equal to the total unadjusted excess growths of the bidders who are behind i in the queue. Finally, Conditions 1 and 5 says that the unadjusted excess growths for bidders who are first in the queue adds up to the given aggregate excess growth Ξ.

An example of a balanced division is $\Xi_i(x) = \Xi(x)/N$, and

$$\Xi_{(z,i,j)}(x_{N(z,i)}) = \frac{1}{|N(z,i)|} \sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)})\rho_i(x_i).$$

Another example is $\Xi_1(x) = \Xi(x)$, and

$$\Xi_{(z,i,j)} = \begin{cases} \sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)})\rho_i(x_i) & \text{if } j = i+1; \\ 0 & \text{otherwise,} \end{cases}$$

where $i+1$ is defined in modulo arithmetic, i.e., $N+1 = 1$.

Given a balanced division $\{\Xi_z\}_{z \in Z}$, we can define the individual excess growths

$$\xi_i(x) = \sum_{z \in Z(i)} \left(\Xi_{(z,i)}(x_i, x_{N(z,i)}) - \sum_{y_i \in X_i} \Xi_{(z,i)}(y_i, x_{N(z,i)})\rho_i(y_i) \right). \tag{37}$$

We claim that ξ_i gives participation secure transfers. This can be verified by checking:

$$\begin{aligned} \sum_{x_i \in X_i} \xi_i(x)\rho_i(x_i) &= \sum_{z \in Z(i)} \left(\underbrace{\sum_{x_i \in X_i} \Xi_{(z,i)}(x_i, x_{N(z,i)})\rho_i(x_i) - \sum_{y_i \in X_i} \Xi_{(z,i)}(y_i, x_{N(z,i)})\rho_i(y_i)}_{=0} \right) \\ &= 0. \end{aligned}$$

Moreover, $\{\xi_i\}_{1 \leqslant i \leqslant N}$ has aggregate excess growth Ξ, since

$$\begin{aligned} \sum_{i=1}^{N} \xi_i(x) &= \sum_{i=1}^{N} \Xi_i(x) \\ &\quad + \sum_{i=1}^{N} \sum_{\{z \in Z(i) : |z| < N-1\}} \left[\underbrace{\sum_{j \in N(z,i)} \Xi_{(z,i,j)}(x_{N(z,i)}) - \sum_{y_i \in X_i} \Xi_{(z,i)}(y_i, x_{N(z,i)})\rho_i(y_i)}_{=0} \right] \\ &\quad - \sum_{i=1}^{N} \sum_{\{z \in Z(i) : |z| = N-1\}} \underbrace{\sum_{y_i \in X_i} \Xi_{(z,i)}(y_i)\rho_i(y_i)}_{=0} \\ &= \Xi(x). \end{aligned}$$

In this last calculation, all we have done is to break up the sum over sequences into those that end in i and those that end in (i,j), then we use the fact that $N(z,i) = \varnothing$ when $|z| = N-1$, so the balanced condition implies that the terms in the third line are all zero.

Given an arbitrary profile of participation secure transfers, we can "recover" their individual excess growths ξ from equation (37) with the balanced division $\{\Xi_z\}_{z \in Z}$ such that $\Xi_\varnothing = \Sigma\xi$, $\Xi_i = \xi_i$ and $\Xi_z = 0$ for all other z. Thus, the participation secure transfers given by the balanced divisions are complete.

When $N = 2$, then Z just consists of $\{\varnothing, 1, 2, 12, 21\}$. For a balanced division $\{\Xi_z\}_{z \in Z}$, Ξ_1 and Ξ_2 are balanced and satisfies $\Xi = \Xi_1 + \Xi_2$; moreover, we have

$$\sum_{x_2 \in X_2} \Xi_2(x_1, x_2)\rho_2(x_2) = \Xi_{21}(x_1),$$

and likewise for Ξ_1 and Ξ_{12}. Thus, the expression of ξ_i in (37) reduces to equation (25).

B.3 Transfer rules for constant-sum values

In this section we compute the transfers (applying Propositions 1 and 2) for the constant sum example (Section 4). We first take $k \to \infty$ and then take $m \to 0$.

B.3.1 $\Xi_i(x) = \Xi(x)/2$

Let us first consider the case where the bidders initially get half of the aggregate excess growth: $\Xi_1 = \Xi_2 = \Xi/2$.

Taking $k \to \infty$, the aggregate excess growth from equation (27) becomes

$$\Xi(x) = -\frac{1}{2m}(1 - \exp(-m)) + \frac{1}{2m}\mathbb{I}_{|x_1 - x_2| < m},$$

where we have modified the constant to make Ξ balanced for a fixed $m > 0$ (see the paragraph following (27) for a discussion about the distribution of $|x_1 - x_2|$).

The individual excess growth (equation (25)) and the transfers (equation (22)) in the limit are

$$\begin{aligned}
\xi_i(x) =& \frac{1}{2}\Xi(x) - \frac{1}{2}\int_{y_i=0}^{\infty} \exp(-y_i)\Xi(y_i, x_j)dy_i + \frac{1}{2}\int_{y_j=0}^{\infty} \exp(-y_j)\Xi(x_i, y_j)dy_j \\
=& -\frac{1}{4m}(1 - \exp(-m)) + \frac{1}{4m}\mathbb{I}_{|x_i - x_j| < m} \\
& - \frac{1}{4m}\int_{y_i=0}^{\infty} \exp(-y_i)\mathbb{I}_{|y_i - x_j| < m}dy_i + \frac{1}{4m}\int_{y_j=0}^{\infty} \exp(-y_j)\mathbb{I}_{|x_i - y_j| < m}dy_j
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
t_i(x) =& \frac{-1}{\exp(-x_i)}\int_{y_i=x_i}^{\infty} \exp(-y_i)\xi_i(y_i, x_j)dy_i \\
=& -\int_{y_i=0}^{\infty} \exp(-y_i)\xi_i(x_i + y_i, x_j)dy_i.
\end{aligned}$$

Substituting ξ_i into t_i, we get

$$\begin{aligned}
t_i(x) =& \frac{1}{4m}(1 - \exp(-m)) - \int_{y_i=0}^{\infty} \frac{1}{4m}\mathbb{I}_{|x_i + y_i - x_j| < m} \exp(-y_i)dy_i \\
& + \frac{1}{4m}\int_{y_i=0}^{\infty} \exp(-y_i)\mathbb{I}_{|y_i - x_j| < m}dy_i - \frac{1}{4m}\int_{y_i=0}^{\infty}\int_{y_j=0}^{\infty} \exp(-y_j - y_i)\mathbb{I}_{|x_i + y_i - y_j| < m}dy_j dy_i \\
=& \underbrace{\frac{1}{4m}(1 - \exp(-m))}_{A} - \underbrace{\frac{1}{4m}(\exp(-\max(x_j - x_i - m, 0)) - \exp(-\max(x_j - x_i + m, 0)))}_{B} \\
& + \underbrace{\frac{1}{4m}(\exp(-\max(x_j - m, 0)) - \exp(-(x_j + m)))}_{C} - \underbrace{\frac{1}{4m}(L(-x_i + m) - L(-x_i - m))}_{D},
\end{aligned} \tag{39}$$

where L is the CDF of a Laplace distribution:

$$L(z) = \begin{cases} \exp(z)/2 & z < 0; \\ 1 - \exp(-z)/2 & z \geqslant 0. \end{cases} \tag{40}$$

Using L'Hôpital's rule to take $m \to 0$, we get

$$\begin{aligned} A &= 1/4; \\ B &= \begin{cases} 0 & x_j < x_i; \\ 1/4 & x_j = x_i; \\ \exp(-x_j + x_i)/2 & x_j > x_i; \end{cases} \\ C &= \begin{cases} 1/4 & x_j = 0; \\ \exp(-x_j)/2 & x_j > 0; \end{cases} \\ D &= \exp(-x_i)/4. \end{aligned}$$

So in the limit as $m \to 0$, we have

$$t_i(x) = \begin{cases} 1/4 - \exp(-x_j + x_i)/2 + \exp(-x_j)/2 - \exp(-x_i)/4 & x_i < x_j; \\ \exp(-x_j)/2 - \exp(-x_i)/4 & x_j = x_i; \\ 1/4 + \exp(-x_j)/2 - \exp(-x_i)/4 & x_i > x_j, \end{cases} \tag{41}$$

when $x_j > 0$, and

$$t_i(x) = \begin{cases} 0 & x_i = 0; \\ 1/2 - \exp(-x_i)/4 & x_i > 0, \end{cases} \tag{42}$$

when $x_j = 0$.

B.3.2 $\Xi_i(x) = \Xi(x)/2 + c\,\mathrm{sign}(x_i - x_j)$

We now show that an alternative initial division of the aggregate excess growth yields a simpler transfer rule. To this end suppose initially bidder i gets $\Xi_i(x) = \Xi(x)/2 + c\,\mathrm{sign}(x_i - x_j)$, where c is a constant, and $\mathrm{sign}(z)$ is 1 if $z > 0$, -1 if $z < 0$, and zero if $z = 0$.

The individual excess growth (equation (25)) in the limit is now

$$\begin{aligned} \xi_i(x) =& \frac{\Xi(x)}{2} + c\,\mathrm{sign}(x_i - x_j) - \int_{y_i=0}^{\infty} \exp(-y_i) \left(\frac{\Xi(y_i, x_j)}{2} + c\,\mathrm{sign}(y_i - x_j) \right) dy_i \\ &+ \int_{y_j=0}^{\infty} \exp(-y_j) \left(\frac{\Xi(x_i, y_j)}{2} + c\,\mathrm{sign}(y_j - x_i) \right) dy_j \end{aligned}$$

Let us denote the difference between the above equation and equation (38) as

$$\eta_i(x) \equiv c\,\mathrm{sign}(x_i - x_j) - \int_{y_i=0}^{\infty} \exp(-y_i) c\,\mathrm{sign}(y_i - x_j) dy_i + \int_{y_j=0}^{\infty} \exp(-y_j) c\,\mathrm{sign}(y_j - x_i) dy_j.$$

The transfer given by $\Xi_i(x) = \Xi(x)/2 + c\,\mathrm{sign}(x_i - x_j)$ is equation (39) plus

$$\begin{aligned}
\Delta_i(x) \equiv &- \int_{y_i=0}^{\infty} \exp(-y_i)\eta_i(x_i + y_i, x_j)dy_i \\
= &- \int_{y_i=0}^{\infty} c\,\mathrm{sign}(x_i + y_i - x_j)\exp(-y_i)dy_i + \int_{y_i=0}^{\infty} \exp(-y_i)c\,\mathrm{sign}(y_i - x_j)dy_i \\
&- \int_{y_i=0}^{\infty}\int_{y_j=0}^{\infty} \exp(-y_j - y_i)c\,\mathrm{sign}(y_j - x_i - y_i)dy_j dy_i \\
= &- c(2\exp(-\max(x_j - x_i, 0)) - 1) + c(2\exp(-x_j) - 1) + c(1 - \exp(-x_i)).
\end{aligned}$$

Taking $c = -1/4$ and adding $\Delta_i(x)$ to equations (41) and (42), we get as $m \to 0$

$$t_i(x) = \begin{cases} 0 & x_i < x_j; \\ 1/4 & x_j = x_i; \\ 1/2 & x_i > x_j, \end{cases}$$

when $x_j > 0$, and

$$t_i(x) = \begin{cases} 0 & x_i = 0; \\ 1/4 & x_i > 0, \end{cases}$$

when $x_j = 0$.