The Impact of Valuation Heterogeneity and Network Structure on Equilibrium Prices in Supply Chain Networks

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Abstract
This paper studies bargaining in two-sided supply chain networks where manufacturers on the demand side purchase an input from suppliers on the supply side. The manufacturers may have heterogeneous valuations on the input sold by the suppliers. In such a supply chain network, a manufacturer and a supplier must have a business relationship or "link" to bargain and trade with each other. However, a firm on one side of the supply chain network might not have a business relationship with every firm on the other side of the supply chain network. We show that valuation heterogeneity, supply and demand balance, and network structure are the main factors that influence the equilibrium prices, trading pattern and surplus allocation in such a supply chain network. Valuation heterogeneity among manufacturers can mitigate unfavorable supply and demand balance to protect some surplus for the manufacturers and leads to higher price dispersion in the supply chain network. We demonstrate that bargaining effectively takes place in smaller subnetworks in a general supply chain network and develop a decomposition algorithm to decompose the general network into these smaller subnetworks, which simplifies the analysis of the general supply chain network significantly. We then identify what types of supply chain networks are competitive so that all trades happen at the same competitive price defined solely according to the aggregate supply and demand balance in the supply chain network and/or efficient so that only manufacturers with the highest valuations are able to trade. We also identify what types of links can be added into a supply chain network to improve its competitiveness and/or efficiency.

1 Introduction and Related Literature
This paper considers two-sided supply chain networks where manufacturers on the demand side purchase an intermediate input or component (e.g., DRAM chips) for their final products (e.g.,
tablets and/or smart phones) from suppliers on the supply side. A manufacturer and a supplier in such a supply chain network must have a business relationship or “link” to bargain and trade with each other. However, a manufacturer may not have a business relationship with every supplier in the supply chain network. Lacking a business relationship between two firms can be caused by various factors such as transportation costs, geopolitical restrictions, technological compatibility, social contacts [among managers], etc. (Jackson, 2008). The pattern of the business relationships between manufacturers and suppliers in a supply chain forms the network structure of the supply chain. The network structure of a supply chain restricts and influences competition, and therefore the possible trading outcomes in the supply chain network. Two firms on different sides of a supply chain network cannot trade if they have no business relationship with each other. Firms on one side of a supply chain network may need to compete for the trade with a firm on the other side of the network they all have a business relationship with.

Although they are buyers of a common input in a supply chain network, manufacturers can be different from each other. First, their final products can be different: The end products could be totally different products or they could be similar products that have different values and margins and target different segments or markets. As a result, the manufacturers’ valuations (or willingness to pay) on the component are different. Second, the manufacturers’ business relationships with the suppliers in the network could be different too. The manufacturers may have business relationships with different suppliers in the network and some of them have business relationships with more suppliers than others do. In the presence of these differences in such a supply chain network, many interesting questions, from both firms’ perspectives, and the whole supply chain network’s perspective, arise. How should a manufacturer in a supply chain network bargain with suppliers who have a business relationship with her to secure the supply of the component for her final products at a good price? How should a supplier respond to offers from manufacturers? When market clears, what does the final trading outcome of the network look like, i.e., who will trade with whom in the network at what prices? Is the final trading outcome competitive so that all trades happen at the competitive price determined by the total supply and demand in the network? Is the final trading outcome efficient so that only manufacturers with the highest valuations on the component are able to trade? How can the competitiveness, as well as the efficiency, of a supply chain network be improved? This paper aims to answer these questions.

We model a two-sided supply chain network via a bipartite graph. The nodes in the bipartite graph represent firms: The manufacturers are on one side and the suppliers are on the other side of the bipartite graph. The link between two firms on different sides of the graph represents that the two firms have a business relationship and can bargain and trade with each other. The manufacturers have different valuations on the component. To facilitate tractability, we assume that
each manufacturer purchases only one unit of the component and each supplier can supply only one unit of the component. Therefore, in the supply chain network, one manufacturer and one supplier can form a match with balanced supply and demand of the component. The manufacturers bargain with the suppliers on the prices of the component following a generalized version of Rubinstein (1982) bargaining model where bargaining occurs in an alternating order between the manufacturers and the suppliers. Without loss of generality, as buyers in the network, the manufacturers initiate the bargaining process by simultaneously offering prices that they are willing to accept for the component to the suppliers with whom they have a link. Then, the suppliers simultaneously respond to offers they received by deciding whether to accept one of the offers or reject all offers. If a supplier accepts an offer from a manufacturer, they will trade at the offered price and leave the network. Next, all remaining suppliers, who rejected all offers they received, will propose counter-offers to the remaining manufacturers with whom they are linked. Then, the remaining manufacturers will respond to the counter-offers. Firms trade if offers are accepted and leave the network. Remaining firms will keep bargaining by proposing and responding to offers in this alternating order until all firms trade or there are no links left. With this model of supply chain network, we establish the following main findings.

First, we show that valuation heterogeneity among manufacturers plays an important role in interacting with supply and demand balance and network structure of a supply chain network to influence the equilibrium outcome of the supply chain network. Existing literature (e.g., Corominas-Bosch 2004) has shown that if there are more manufacturers than suppliers (i.e., there is a supply shortage and manufacturers need to compete for limited supplies) and the manufacturers have the same valuation on the component in a supply chain network, the suppliers would be able to take the advantage of the competition among the manufacturers to push the prices all the way up to their homogenous valuation. As a result, the suppliers capture all the surpluses of the trades while the manufacturers get zero surplus. However, we find that this is no longer the case when the manufacturers have heterogeneous valuations. With heterogenous manufacturer valuations, the suppliers cannot simply push the prices to a uniform level to capture all surpluses of the trades any more because the manufacturers are now differentiated from each other in their valuations. Therefore, valuation heterogeneity serves as a counter force to unfavorable supply and demand imbalance to protect the manufacturers from price hikes and surplus extractions by the suppliers. Furthermore, when manufacturers have heterogeneous valuations, suppliers prefer to trade with manufacturers with high valuations because they can potentially pay more. In other words, suppliers have more preferences over whom to trade with than they do when manufacturers have the same valuation. This increased preference leads to more prices or price dispersion in a supply chain network because the exact price a supplier can get from a trade critically depends on
the exact valuation of the manufacturer involved in the trade.

Second, we find that bargaining actually takes place in smaller subnetworks in a general supply chain network because firms may selectively bargain with a subset of firms with whom they are linked. In other words, firms may ignore some of their links in the network that are not useful for them to get a better deal. We develop a decomposition algorithm to identify these subnetworks in which bargaining between firms will effectively take place in general supply chain networks. The decomposition algorithm generalizes existing network decomposition algorithms by explicitly taking the valuation heterogeneity among manufacturers into account. We show that any supply chain network can be decomposed into a union of three types of subnetworks. In each of these three types of subnetwork, the equilibrium prices of all trades are determined jointly by manufacturer valuations, supply and demand balance, and network structure in the subnetwork independent of the rest of the network. With this decomposition algorithm, we can simplify the analysis of a general supply chain network significantly to determine the equilibrium outcome of the network.

Third, we identify what types of supply chain networks are competitive and/or efficient. In a competitive supply chain network, all trades must happen at the single competitive price which is defined purely by the aggregate supply and demand balance in the network without any restriction of the network structure. For a supply chain network to be competitive, it should have enough links between the manufacturers and the suppliers in the network so the supply and the demand of the network can be matched freely to achieve the competitive price. When manufacturers have different valuations, efficiency of a supply chain network becomes a concern. In an efficient supply chain network, only the manufacturers with the highest valuations can trade so the total surpluses of all trades are maximized (which is not a concern when all manufacturers have the same valuation). For a supply chain network to be efficient, we find that manufacturers with high valuations must be well connected, i.e., have more than one link, so they can secure a trade. We show that a competitive supply chain network must be efficient, but the opposite may not be true. In other words, the conditions for a supply chain network to be competitive are more restrictive than the ones for it to be efficient. These results suggest that there could exist a divergency between competitiveness and efficiency of a supply chain network.

Fourth, we identify the types of links to add into a supply chain network to improve the competitiveness and/or efficiency of the network. Adding a new link into a supply chain network can improve its competitiveness if the two firms connected by the link can do better in the bargaining with the link than without the link. Then, they will appreciate and utilize this new link in their bargaining. As a result, supply and demand in the network has one more link to be matched in the bargaining process, which improves the competitiveness of the supply chain network. Adding a new link into a supply chain network can improve its efficiency if with the new link more trades
can happen in the network or manufactures with higher valuations can replace manufacturers with lower valuations to trade in the network. Therefore, an efficiency improving link must connect to a manufacturer, especially with a high valuation, in a subnetwork with more manufacturers than suppliers where the manufacturers need to compete for limited supply. With this new link, the manufacturer can potentially reach out to a new source of supply outside of the subnetwork to either make a new trade possible or replace a manufacturer with lower valuation to trade to improve the efficiency of the supply chain network. Reflecting the divergency between competitiveness and efficiency of a supply chain network, we show that there exist links that can improve either competitiveness or efficiency of a supply chain network, but not both.

Existing supply chain literature mainly focuses on simple supply chains with limited numbers of firms and specific network structures such as one-manufacturer-one-supplier (e.g., Spengler, 1950; Pasternack, 1985), one-manufacturer-multiple-suppliers (e.g., Bernstein and Federgruen, 2005; Cachon and Lariviere, 2005; Netessine and Zhang, 2005) and competing supply chains each with one-manufacturer-one-supplier (e.g., Ha and Tong 2008, Wu and Chen 2010). However, much more complex supply chains are present in practice (Netessine, 2011). Our paper takes the first step to fill the gap between existing supply chain literature and practice by studying complex supply chains with multiple manufacturers and suppliers and general network structures between them. Two related papers are Corbett and Karmarkar (2001) and Adida and DeMiguel (2011) who study supply chains with multiple firms at each level of the supply chains. However, both papers consider a special network structure with perfect competition, which essentially assumes that each firm on one side of a supply chain has a link with every firm on the other side of the supply chain. Loss of efficiency caused by competition in supply chains measured by price of anarchy, defined as the largest ratio of profits between the coordinated supply chain and the decentralized supply chain, has been studied in recent (e.g., Martinez de Albéniz and Simchi-Levi 2003, Parakis 2007, Parakis and Roels 2007, Kluberg and Parakis 2012). These studies normally consider specific supply chain structures such as assembly structure or distribution structure. There are papers studying bargaining in supply chains with specific structures. Nagarajan and Bassok (2008) and Nagarajan and Sosic (2009) study bargaining models in assembly systems where one buyer bargains with several suppliers. Lovejoy (2010) considers bargaining in multi-tier supply chains where only one firm in each tier will emerge as the supplier for the downstream tier. The key difference between our paper and the above papers is that our model considers general network structures without making specific assumptions about the pattern of relationships between firms on the two sides of a supply chain network.

Also related is the recent research stream on social networks and supply networks. Acemoglu, Dahleh, Lobel and Ozdaglar (2011) study learning in social networks and Lobel and Sadler (2014)
study information diffusion through social learning in social networks. Candogan, Bimpikis and Ozdaglar (2012) consider a pricing optimization problem of a monopoly selling a good to consumers in a social network where a consumer’s usage level of the good depends on the usage of her neighbors in the network. Instead of studying social networks where consumers are connected, we consider two-sided supply chain networks with buyers on one side and sellers on the other side of the network. Chou, Chua, Teo, and Zheng (2010) studies the benefits of long chain and sparse process structure for a single firm. A supply chain network formation problem is studied by Bimpikis, Fearing and Tahbaz-Salehi (2014). They found that multi-sourcing by upstream firms may lead to a intertwined supply chain where the likelihood of simultaneous disruptions to the downstream firms increases. We do not consider network structure design or formation as these papers, but focus on the bargaining between buyers and sellers in exogenously given two-sided supply chain networks.

Our paper is also related to economics literature on bargaining in networks. This line of research mostly focuses on centralized trading mechanisms in order to implement efficient matching outcomes. A closely related paper is Corominas-Bosch (2004), which focuses on the characteristics of the competitive network structures. Our model generalizes the Corominas-Bosch (2004) setting by considering that the values of bilateral trades in a network can be different depending on the valuations of the buyers involved in the trades. More recently, Polanski (2007), Manea (2008), and Abreu and Manea (2009) provide intuition on bargaining power of homogeneous firms in various markets with communication restrictions. One key difference between our model and theirs is that we restrict our attention to only two-sided markets while relaxing the homogeneity assumption on firms. Elliot (2012) considers a similar model and focuses on the impact of network structure on the cooperative outcome of the trades and examines the inefficiencies that arise due to the imperfection of the competition. Finally, Kranton and Minehart (2001) focus on the efficient implementation of networks via centralized auction mechanism in a non-strategic sellers environment.

The next section develops the model while introducing notation and preliminary mathematical tools. Section 3 considers small supply chain networks with at most two manufacturers and two suppliers. In Section 4, we consider general supply chain networks and develop the decomposition algorithm to identify subnetworks in which bargaining will effectively take place. Section 5 studies the competitiveness and efficiency of supply chain networks. Section 6 concludes the paper.

2 The Model

Consider a supply chain network with $|S|$ suppliers (sellers) $S = \{s_1, s_2, ..., s_{|S|}\}$ and $|M|$ manufacturers (buyers) $M = \{m_1, m_2, ..., m_{|M|}\}$.\footnote{For any finite set $X$, $|X|$ represents the number of elements in $X$.} Each supplier (he) sells one unit of an identical
component that will be used to assemble a final product by a manufacturer (she). We assume that the component is worthless for a supplier unless he sells the component to a manufacturer. In other words, the suppliers have homogeneous reservation valuation, which is zero, on their components. Each manufacturer needs to buy only one unit of the component from one supplier to assemble one unit of the final product (i.e., single sourcing). Manufacturers have heterogenous valuations (or willingness-to-pays) on the component, which presumably could be determined by the market values of their final products. Let $v_j \in [\underline{v}, \overline{v}]$ denote the valuation of manufacturer $m_j$ on the component, where $\underline{v}$ and $\overline{v}$ are the lowest and highest possible valuations, respectively. Let $\mathbf{v} = (v_1, v_2, ..., v_{|M|})$ denote the profile of the manufacturers’ valuations. Without loss of generality, we label the manufacturers throughout the paper in decreasing order in their valuations $v_j$ for all $m_j \in M$, that is, $v_1 \geq v_2 \geq ... \geq v_{|M|}$.

The supply chain network has existing business relationships between the suppliers and manufacturers that can be represented by a non-directed bipartite graph $G = (S, M, L)$, which consists of a set of nodes formed by the suppliers in $S$ on the supply side and the manufacturers in $M$ on the demand side, together with the set of links in $L$. Each link in $L$ joins a supplier with a manufacturer and can be represented as a subset of the cartesian product of $S$ and $M$, that is $L \subseteq S \times M$. An element of $L$, say a link between supplier $s_i$ and manufacturer $m_j$, is denoted as $ij$. For notational consistency, we always write a supplier first for any link. A link in $L$ is a representation of an established business relationship between the two nodes/firms that it connects. A trade between two firms can only take place if they currently have an established business relationship or are “linked” or “connected.” If there is no link between two firms, it means the two firms have not established a viable business relationship and they cannot trade with each other. The lack of a business relationship between firms in a market could be caused by various reasons such as they are geographically far apart, may not have enough information about each other’s business, or purposely choose to not do business with each other because of high tariffs or exclusive agreements with other firms. Therefore, a supply chain network (or, simply, network) of suppliers and manufacturers can be defined as $(G, \mathbf{v})$. We will abbreviate the notation from $(G, \mathbf{v})$ to $G$ for a supply chain network where manufacturers have homogenous valuations, that is, $v_1 = v_2 = ... = v_{|M|}$.

The suppliers and the manufacturers in the supply chain network bargain with each other to trade according to an alternating multi-period process. In the first period, we assume that the manufacturers take the lead to simultaneously propose the highest prices they are willing to pay for the component to the suppliers they are linked to. The suppliers simultaneously respond by determining whether to accept one of the prices proposed by their linked manufacturers or reject all. A trade is possible between a supplier and a manufacturer only if they are linked in the supply network and the supplier is willing to accept the manufacturer’s proposed price. There
could be multiple feasible trade patterns in each round of bargaining. Then, a surplus maximizing mechanism, which is defined in detail below, determines the effective trade pattern. After a supplier and a manufacturer trade, they leave the game. Firms who could not trade in the first period remain in the game to continue to bargain while preserving the remaining links in the network.

In the second period, remaining suppliers would take the lead to propose the lowest prices they would like to accept for the component, then the remaining manufacturers would respond whether to accept one price proposed by a linked supplier or reject all. A trade is possible between a supplier and a manufacturer in the second period only if they are linked and the manufacturer is willing to accept the supplier’s proposed price. The surplus maximizing mechanism will determine the effective trade pattern if there are multiple feasible trade patterns. After a supplier and a manufacturer trade, they leave the game. Firms who could not trade in the second period remain in the game to continue to bargain while preserving the remaining links in the network. In the third period, remaining manufacturers take the lead to propose prices, and remaining suppliers respond as in the first period. This alternating bargaining process repeats itself until all firms trade or there are no remaining links between the firms left. We are interested in the subgame perfect Nash equilibrium payoffs of this game.

Each firm discounts the future with a common factor $\delta$. If a manufacturer $m_j$ with valuation $v_j$ trades with a supplier at price $p$ in period $t$, the manufacturer $m_j$ receives a payoff of $\delta^t (v_j - p)$ and the supplier, who has zero reservation value, receives a payoff of $\delta^t p$. Thus, the total surplus of this trade is $\delta^t (v_j - p) + \delta^t p = \delta^t v_j$. Therefore, once the price of a trade is determined, the payoffs of the trading partners are determined as well. Thus, throughout the paper, we will focus on the equilibrium prices of the game instead of the equilibrium payoffs of the game, which are directly determined by the equilibrium prices. Recall that $v_j \in [\underline{v}, \overline{v}]$ for all $j \in M$. We carry the following technical assumption throughout the rest of the paper: $\underline{v} \geq \frac{\delta}{1+\delta} \overline{v}$. This assumption essentially ensures that manufacturers’ valuations are reasonably close so there is effective competition among them in the supply chain network.

We now define the mechanism that selects the effective trade pattern when multiple feasible trade patterns exist in a period. In period $t$, the supply chain network is represented by $(G_t, v_t)$, where $G_t = (S_t, M_t, L_t)$ (with $G_1 = G$) and $v_t = (v_j | j \in M_t)$. According to the game described above, if $t$ is odd, then manufacturers propose prices and suppliers respond. If $t$ is even, suppliers propose prices and manufacturers respond. After all prices are proposed and responded in period $t$, we represent the set of firms that can be a part of the effective trade pattern in period $t$ via the subgraph $\tilde{G}_t = (\tilde{S}_t, \tilde{M}_t, \tilde{L}_t)$. A firm is part of the effective trade pattern if and only if there is at least one linked firm that is willing to trade with it at a proposed price. To define the effective trade formally, we need to provide some graph theory concepts. A matching in a graph is a subset of links
such that each node in the graph is connected to at most one link. A matching that covers all the nodes in the graph is a perfect matching and a matching that contains the largest possible number of links is a maximum matching. The set of possible trade patterns in period \( t \) is determined by the set of possible matchings in \((\tilde{G}_t, \tilde{\nu}_t)\). The mechanism that chooses the effective trade pattern from the set of all feasible trade patterns uses a matching in \( \tilde{G}_t \) with the highest total surplus, which is the summation of the valuations of the manufacturers in the selected matching at time \( t \). If there is more than one surplus maximizing matching, the mechanism picks one of them randomly. Notice that this procedure is well-defined because the set of all matchings at any time period is finite.

Because we represent the business relationship using a non-directed bipartite graph, it is useful to introduce the following concepts. A firm that is linked with firm \( k \) in \( G = (S, M, L) \) is called a neighbor of \( k \) and the set of all neighbors of \( k \) is denoted by \( G(k) = \{ i \in S \cup M \mid ki \in L \} \). Furthermore, we denote the neighbors of the set \( \tilde{X} \subseteq X \) as \( G(\tilde{X}) = \bigcup_{k \in \tilde{X}} G(k) \) where \( X \in \{S, M\} \).

A subgraph \( \tilde{G} = (\tilde{S}, \tilde{M}, \tilde{L}) \) of \( G \) is a graph such that \( \tilde{S} \subseteq S, \tilde{M} \subseteq M \), and the restriction of \( L \) over \( \tilde{S} \cup \tilde{M} \), denoted as \( \tilde{L} = L|_{\tilde{S} \cup \tilde{M}} \). A path in a graph is a sequence of nodes such that from each of its nodes there is a link to the next node in the sequence. A bipartite graph is connected if there exists a path linking any two nodes of the graph. Although they are slightly different concepts, throughout the paper, we use the terms “network” and “graph” interchangeably. We only consider supply networks whose business relationship graphs are connected. If the supply network’s business relationship graph is disconnected, we can apply all of our results to each disconnected component of the network separately.

3 Simple Supply Chain Networks

We will analyze several simple supply chain networks with at most two suppliers and two manufacturers. The results about these simple networks can offer helpful insight for understanding larger and more general networks. We start with the cases in which there are at most three firms in the supply chain. Throughout the paper, by uniqueness, we refer to the uniqueness in terms of equilibrium payoffs not equilibrium strategies.

**Proposition 1.** Consider a supply chain network with at most three firms. Then, the unique subgame perfect Nash equilibrium can be characterized as follows:

(i) when one supplier links to one manufacturer with valuation \( v \), the equilibrium price is \( \delta \frac{1}{1+\delta} v \);
(ii) when two suppliers link to one manufacturer with valuation \( v \), the equilibrium price is zero;
(iii) when one supplier links to two manufacturers with homogenous valuation \( v \), the equilibrium price is \( v \);
when one supplier links to two manufacturers with heterogenous valuations \( v_1 > v_2 \) respectively, the equilibrium price is \( v_2 \).

**Proof:** All proofs are provided in the appendix.

The above proposition characterizes the equilibrium prices for all possible supply chain networks with at most three firms. With at most three firms, all bargainings will finish in one period in equilibrium (Rubinstein 1982). Therefore, if a manufacturer with valuation \( v \) trades with a supplier at price \( p \), the manufacturer receives a payoff of \( v - p \) and the supplier receives a payoff of \( p \). Thus, the equilibrium payoffs for the firms in the three supply chain networks are specified by the proposition as well.

In a supply chain network with only one supplier and one manufacturer, the firms engage in the well-known alternating offer bargaining game of Rubinstein (1982), in which the unique equilibrium price is \( \frac{\delta}{1+\delta}v \) as specified in part (i) of the above proposition, which implies that the equilibrium payoffs are \( \frac{1}{1+\delta}v \) for the manufacturer and \( \frac{\delta}{1+\delta}v \) for the supplier.

In a supply chain network with one side shorter (fewer number of firms) than the other side, there is an imbalance between supply (i.e., the number of suppliers) and demand (i.e., the number of manufacturers). When bargaining in such a supply chain network, the firms on the short side of the network, which has less rivals, have an advantage over the firms on the long side, which has more rivals. Parts (ii) and (iii) of the above proposition indicate that when firms on each side of the network have homogenous valuations, the advantage of the short side caused by the imbalance between supply and demand is the dominating force of determining the equilibrium prices, payoffs and allocation of surplus. Part (ii) considers a network with two homogenous suppliers linked to one manufacturer with valuation \( v \). In this case, there is more supply than demand. The sole manufacturer on the short side of the network can take advantage of the competition between two homogenous suppliers on the long side of the network to press the price down to their reservation values, which are all zero. With price at zero, the manufacturer collects all the economic surplus of the trade and receives a payoff of \( v \). According to part (iii), similar observations are true in a network with one supplier linked to two homogeneous manufacturers with valuations \( v \). In this case, there is more demand than supply. The sole supplier on the short side is able to push the price up to \( v \) which is the reservation value for the two homogenous competing manufacturers on the long side. With price at \( v \), the supplier collects all the economic surplus of the trade and receives a payoff of \( v \). Parts (i), (ii) and (iii) are consistent with the findings in Corominas-Bosch (2004).

However, part (iv) of the above proposition indicates that when both manufacturers have heterogenous valuations, the equilibrium is sharply different from the equilibrium specified in part (iii), which has exactly the same network structure. This difference clearly illustrates the potential...
impact of valuation heterogeneity of the manufacturers on the equilibrium outcome in a supply chain network. Specifically, when one supplier is linked to two manufacturers with heterogenous valuations, the supplier on the short side is no longer able to collect all the economic surplus of the trade, which implies that the advantage of the supplier on the short side is limited in the presence of valuation heterogeneity among the manufacturers on the long side. As we discussed above, the advantage of the supplier on the short side of the network would push the price up. However, with heterogenous valuations, the price cannot be pushed all the way up to the valuation of the manufacturer who would eventually win the trade. This is because as the price reaches the lower valuation of the two manufacturers, \( v_2 \), the manufacturer with valuation \( v_2 \) would drop out of the competition and the manufacturer with valuation \( v_1 \) would be the only buyer left in the market to win the trade. Given the assumption \( v \geq \frac{\delta}{1+\delta} \bar{v} \) or specifically \( v_2 \geq \frac{\delta}{1+\delta} v_1 \) in this case (this assumption ensures that the two manufacturers are meaningful competitors), the supplier would prefer to accept the price \( v_2 \) right away than bargaining with the manufacturer with valuation \( v_1 \) further in the next period. In other words, with heterogenous valuations, the manufacturers are differentiated. This degree of differentiation would help the manufacturer with higher valuation to partially offset the pricing advantage of the supplier on the short side. With this protection from differentiation, the manufacturer with higher valuation can enjoy some surplus of the trade and prevent the supplier from collecting all the surplus. Therefore, we can clearly see that valuation heterogeneity between manufacturers on the long side and the advantage of the suppliers on the short side caused by the imbalance between supply and demand are two countering forces that will jointly determine the equilibrium in the supply chain network.

We now consider networks in which there are two suppliers, \( s_1 \) and \( s_2 \) and two manufacturers, \( m_1 \) and \( m_2 \) with valuations \( v_1 > v_2 \). There are three possible connected supply chain network structures, which are shown in Figure 1. In the complete network shown in 1(a), each firm is linked with all firms on the opposite side of the network and firms have symmetric network positions. Figure 1(b) represents the case where manufacturer \( m_1 \) with high valuation is “well” connected with links to both suppliers, while manufacturer \( m_2 \) is only linked to one of the suppliers. Figure 1(c) represents the opposite case where manufacturer \( m_2 \) with low valuation is the one that is well connected with links to both suppliers while manufacturer \( m_1 \) is only linked to one of the suppliers. Notice that in these networks, supply and demand are perfectly balanced because there are equal number of firms on both sides of the networks. The equilibrium outcomes of these networks are summarized in the next proposition.

**Proposition 2.** Consider a supply chain network with two suppliers, \( s_1 \) and \( s_2 \) and two manufacturers, \( m_1 \) and \( m_2 \).
Figure 1: All possible connected supply chain networks with two suppliers and two manufacturers.

(i) If the two manufacturers have homogenous valuations (i.e., \( v_1 = v_2 = v \)), then in all supply chain networks in Figure 1(a), (b) and (c), there exists a unique subgame perfect Nash equilibrium where all firms trade at price \( \frac{\delta}{1+\delta} v \);

(ii) If the two manufacturers have heterogenous valuations (i.e., \( v_1 > v_2 \)), then in the supply chain networks in Figure 1(a) and (b), there exists a unique subgame perfect Nash equilibrium where all firms trade at price \( \frac{\delta}{1+\delta} v_2 \);

(iii) If the two manufacturers have heterogenous valuations (i.e., \( v_1 > v_2 \)), then in the supply chain network in Figure 1(c), there exists a unique subgame perfect Nash equilibrium where supplier \( s_1 \) trades with manufacturer \( m_1 \) at price \( \frac{\delta}{1+\delta} v_1 \) and supplier \( s_2 \) trades with manufacturer \( m_2 \) at price \( \frac{\delta}{1+\delta} v_2 \).

In a supply chain network with two suppliers and two manufacturers, because supply and demand are perfectly balanced, firms on one side of the network would not have an advantage over the ones on the other side and all firms will be able to trade. Part (i) of Proposition 2 basically says that when the manufacturers have homogenous valuation, network structure would not affect equilibrium prices at all because all three possible networks shown in Figure 1(a), (b) and (c) are essentially equivalent, in price and payoff terms, to two independent linked pairs whose equilibrium would be determined according to the Rubinstein bargaining game as specified in part (i) of Proposition 1 (see Corominas-Bosch (2004) for detailed discussions of the intuition of this result). Intuitively, since all suppliers and manufacturers are homogenous in this case, they are really indifferent about who to trade with.

If the two manufacturers have heterogenous valuations, we have known from part (iv) of Proposition 1 that the valuation heterogeneity can allow the high valuation manufacturer to press down the price and collect some surplus in a supply chain with three firms. Parts (ii) and (iii) of Proposition 2 indicate that whether the valuation heterogeneity can protect the high valuation manufacturer
in a supply chain with four firms critically depending on whether the high valuation manufacturer is well connected (i.e., has more than one link) in the network or not. Specifically, in the supply chain networks in 1(a) and (b), the high valuation manufacturer, $m_1$ is well connected because she is linked to both suppliers in the network. So, the high valuation manufacturer, $m_1$ is able to trade at price $\frac{\delta}{1+\delta}v_2$, which is lower than the price $(\frac{\delta}{1+\delta}v_1)$ she would pay if she bargains with a supplier independently. In contrast, in the supply chain network in 1(c), the high valuation manufacturer, $m_1$ is not well connected because she is linked to only one supplier, $s_1$. As a result, supplier $s_1$ will strategically isolate manufacturer $m_1$ to engage a Rubinstein bargaining game and trade with her at a higher price $\frac{\delta}{1+\delta}v_1$. Therefore, unlike the cases in part (i) of Proposition 2, with heterogenous manufacturer valuations, the network structure does affect equilibrium prices and payoffs.

The sharp contrast between the equilibrium of two similar networks in Figure 1(b) and 1(c) allows us to see clearly why network structure plays an important role on influencing equilibrium prices and payoffs when manufacturers have heterogenous valuations. In the network in Figure 1(b), given manufacturer $m_1$’s connections and higher valuation, both suppliers prefer to negotiate with $m_1$ simply because $m_1$ could be willing to pay more. Because of this competition between the two suppliers, $m_1$ is able to press the price down to supplier $s_2$’s “reservation” price or outside option $\frac{\delta}{1+\delta}v_2$, which is the price $s_2$ can get by bargaining with manufacturer $m_2$ independently. In the network in Figure 1(c), however, manufacturer $m_1$ is only linked to supplier $s_1$. The only option for $m_1$ is to engage in a Rubinstein bargaining game with $s_1$ and trade at price $\frac{\delta}{1+\delta}v_1$. Supplier $s_1$ is also linked to manufacturer $m_2$, but the highest price $s_1$ can get by negotiating with $m_2$ is $\frac{\delta}{1+\delta}v_2$, which is less than the price $\frac{\delta}{1+\delta}v_1$ he can get by negotiating with $m_1$ under the assumption that $v_1 > v_2$. Thus, $s_1$ prefers to ignore his link with $m_2$ to engage in a Rubinstein bargaining game with $m_1$ in isolation. Being isolated in a one-to-one bargaining, manufacturer $m_1$ will not be able to take advantage of valuation heterogeneity or differentiation as she could, for example, in the network specified in part (iv) of Proposition 1. Hence, bargainings in the network in Figure 1(c) effectively take place independently in two smaller subnetworks between $m_1$ and $s_1$ and between $m_2$ and $s_2$ respectively. Note that in this case we end up with two different prices in equilibrium. Recall that we had only one equilibrium price for the network in Figure 1(c) when manufacturers have the same valuation according to Part (i) of Proposition 2. So, when manufacturers have heterogeneous valuations, suppliers’ preferences to trade with manufacturers with high valuations lead to more prices or price dispersion in a supply chain network because the exact price a supplier can get from a trade critically depends on the exact valuation of the manufacturer involved in the trade.

The results about the simple supply chain networks in Proposition 1 and 2 reveal the rich dynamics among three fundamental factors: the balance between supply and demand, valuation heterogeneity, and network structure on jointly determining equilibrium prices and payoffs in a
supply chain network. When the manufacturers have homogenous valuation, the balance between supply and demand would be the dominating factor that determines the equilibrium. However, when the manufacturers have heterogenous valuations, valuation heterogeneity can potentially act as a countering force to limit the pricing power of the suppliers in supply chains with less suppliers than manufacturers. In addition, valuation heterogeneity makes network structure, specifically how manufacturers with higher valuations are connected in a network, relevant in determining the equilibrium. Furthermore, we have also learned that not all links in a supply chain network are relevant because firms could strategically ignore some of them. Bargaining in a supply chain network will likely happen in smaller subnetworks where the three fundamental factors will determine the equilibrium for each of them. Therefore, to determine the equilibrium in a more complex supply chain network, it is important to identify or decompose the network into subnetworks where bargainings among firms will effectively take place in equilibrium. In the next section, we carry these results and insight into more general supply chain networks.

4 General Supply Chain Networks

We will examine general supply chain networks in this section. To study general supply chain networks, we first need to introduce a set of technical concepts from graph theory.

4.1 Definitions

Let $G = (S, M, L)$ be the underlying bipartite graph of a supply chain network $(G, v)$.

**Definition 1.** A bipartite graph $G$ is non-deficient if and only if $|N(S')| \geq |S'|$ for any subset $S' \subseteq S$ of size $|S'| \leq |M|$ or $|N(M')| \geq |M'|$ for any subset $M' \subseteq M$ of size $|M'| \leq |S|$.

Non-deficiency requires that every subset of firms on the long side of a network has enough neighbors to trade. Figure 2 provides a network that satisfies non-deficiency requirement in (a) and a network that does not in (b). Notice that manufacturers are on the long sides of both

![Figure 2: Non-deficiency of graphs.](image-url)
networks. The network in (a) is non-deficient because in any subsets (i.e., \{m_1\}, \{m_2\}, \{m_3\}, \{m_1, m_2\}, \{m_1, m_3\}, and \{m_2, m_3\}) of the long side of the network \(M\), the manufacturers with no more than two manufacturers (|\(S\)| = 2) are all linked to at least the same number of suppliers so that all manufacturers are guaranteed to trade. The network in (b) does not satisfy the non-deficiency requirement since the subset \{m_1, m_2\} is collectively linked to only one supplier \{s_1\} which implies either \(m_1\) or \(m_2\) will not be able to trade. Notice that non-deficiency is defined for the long side of a bipartite graph and does not imply that both sides of the graph have to satisfy the condition.

Because the valuations of the manufacturers are different in our setting, the following definitions of manufacturer types play a crucial role in our results. Let \(N_G^+(u)\) be the neighbors of \(v \in S \cup M\) who have more than one connection in network \(G\); that is, \(N_G^+(u) = \{w \in N_G(u) : |N_G(w)| > 1\}\).

**Definition 2.** Let \(G\) be a non-deficient bipartite graph that contains a perfect matching. A manufacturer \(m_i \in M\) is a “moderate manufacturer” if

(i) \(|N_G(m_i)| = 1\) or,

(ii) \(|N_G(m_i)| > 1\) and \(v_i = \min \{v_j | m_j \in C_G(m_i)\}\) where \(C_G(m_i) = \bigcap_{s \in N_G^+(m_i)} N_G(s)\).

Moderate manufacturers are defined in a bipartite graph that contains a perfect matching or has equal number of firms on both sides. The definition says that a manufacturer is moderate if she is connected to only one supplier or she has the lowest valuation among all her competing manufacturers. The set of moderate manufacturers in a supply chain network is always non-empty since, in any bipartite graph with a perfect matching, the manufacturer with lowest valuation is by definition a moderate manufacturer. Moderate manufacturers in a supply chain network are basically disadvantageous manufacturers in terms of either connection or valuation or both. Figure 3 identifies the moderate manufacturers for the supply chain networks in Proposition 2. Notice that the networks in Figure 3(a) and (b) have only one moderate manufacturer \(m_2\) who has the lowest valuation as compared to her competitor \(m_1\) for \(s_1\) in (a) and has only one link in (b), whereas the
network in Figure 3(c) has two moderate manufacturers $m_1$ who has only one link, and $m_2$ who has the lowest valuation as compared to her competitor $m_1$ for $s_1$. As we will see in the next section, this difference will play a crucial role in determining the equilibrium strategies of the firms.

Although moderate manufacturers are disadvantageous to their rivals due to their limited connections and/or low valuations, they can still trade at the end of the bargaining process because of the balanced supply and demand in a network with perfect matching. However, in networks with more manufacturers than suppliers or more demand than supply, there is no perfect matching anymore. In such networks, some manufacturers with low valuations, contrary to moderate manufacturers, will not be able to trade due to the scarcity of suppliers. We need to differentiate such manufacturers from moderate manufacturers.

**Definition 3.** Let $G$ be a non-deficient bipartite graph with manufacturer side is longer; i.e., $|M| > |S|$. A manufacturer $m_i \in M$ is a “soft manufacturer” if $v_i \leq v_s$ where $v_s$ is the $(|S| + 1)$th highest valuation in $G$. Furthermore, the “critical soft manufacturer” of $G$ is the manufacturer with the valuation $v_s$, which is denoted by $m_s$.

The main difference between soft and moderate manufacturers is that moderate manufacturers, although disadvantageous, are able to trade whereas soft manufacturers are not. Figure 4 shows a network with two suppliers and five manufacturers. Suppose that the manufacturer valuations are ranked as $v_1 > \ldots > v_5$. Then, the manufacturers $m_3$, $m_4$, and $m_5$ are all soft buyers since they have valuations less than $v_s$, which is equal to $v_3$ because $|S| + 1 = 3$ in this case. Furthermore, $m_3$ is the critical soft buyer since he has (exactly) the third $(|S| + 1)$ highest valuation among all the manufacturers in the network. In supply chains where there are more suppliers than manufacturers, none of the manufacturers are moderate or soft since all manufacturers are on the short side of the network. We are now ready to define three types of supply chain networks.

**Definition 4.** A supply chain network $(G, \mathbf{v})$ is

- (i) type $(G, \mathbf{v})^S$ if $|S| > |M|$ and $G$ is non-deficient;

- (ii) type $(G, \mathbf{v})_k^E$ if $G$ contains a perfect matching and $m_k$ is the only moderate manufacturer in $G$;
(iii) type $(G, v)_s^M$ if $|M| > |S|$ and $G$ is non-deficient with the critical soft manufacturer $m_s$.

In addition, we say that the underlying graph of a supply chain network is $G^S$, $G^E_k$, or $G^M_s$ type if it belongs to a network that is of type $(G, v)^S$, $(G, v)^E_k$, and $(G, v)^M_s$, respectively. It is easy to see that not every graph is one of these types. For example, Figure 2(b) is such a network. To see why, notice that in Figure 2(b) the number of manufacturers is higher than the number of suppliers, the network can only be $(G, v)^M_s$ type and we only need to check the condition (iii) of the definition. As we have already identified that this network does not satisfy the non-deficiency condition ($\{m_1, m_2\}$ are linked with $\{s_1\}$ only), this graph is not a $G^M_s$ type either.

### 4.2 The Decomposition of General Supply Chain Networks

One insight we learned from the analysis of small supply chain networks was that not all links in a supply chain network are relevant (because firms could strategically ignore some of the links) and bargaining in a supply chain network will happen in smaller subnetworks in equilibrium. This insight implies that we can significantly simplify the analysis for general and more complex supply chain networks by focusing on smaller subnetworks of a general supply chain network. We now provide a network decomposition algorithm, which allows us to identify the subnetworks of a general supply chain network where bargaining will take place in equilibrium. The decomposition we will describe below has close connections with the canonical structure theorem of Gallai and Edmonds, which shows a unique decomposition of any graph into three set of subgraphs.\(^2\)

In a setting with homogenous buyers, Corominas-Bosch (2004) shows that any bipartite graph can be decomposed into a union of $G^S$, $G^E$, and $G^M$ types of subgraphs and some extra links by using a simple algorithm (henceforth, CB-algorithm).\(^3\) The CB-algorithm provides a decomposition for a graph that is unique up to the following degree: If a firm belongs to a subgraph of a certain type in a decomposition, then she belongs to the same type of subgraph in every decomposition. This degree of uniqueness is not sufficient when the manufacturers have different valuations since the algorithm does not take their valuations into account. Figure 5 provides an example in which the location of high valuation manufacturer matters. Because the valuations of the manufacturers are different, the price $m_1$ has to pay changes depending on the subgraph that the algorithm puts her in since the valuations of his competitors change. Moreover, as the manufacturer with the highest value, $m_1$ engages in trade no matter which subgraph she ends in and therefore the set of manufacturers who are able to trade is affected by the structure of decomposition. In particular, if $m_1$ joins subnetwork $G^*$ after decomposition, the set of manufacturers who are able to trade would

\(^2\)See Lővász and Plummer (1986), for more details on the Gallai-Edmonds decomposition of graphs.

\(^3\)See the appendix of Corominas-Bosch (2004) for a very detailed description of the algorithm.
Figure 5: A $G^M$ type of subgraph in which the non-uniqueness of the CB-algorithm matters.

be $\{m_1, m_4\}$ (because $m_1$ has the highest valuation in $G^*$ and will win trade with $s_1$, while $m_4$ has the highest valuation in $G^{**}$ and will win trade with $s_2$), but if $m_1$ joins subnetwork $G^{**}$, the set of manufacturers who are able to trade would be $\{m_1, m_2\}$ (because now $m_2$ has the highest valuation in $G^*$ and will win trade with $s_1$, while $m_1$ has the highest valuation in $G^{**}$ and will win trade with $s_2$).

The decomposition algorithm we propose is different from the earlier algorithms since it takes the manufacturer valuations into account. Our algorithm can be summarized as follows. First, the algorithm checks all possible subgraphs for the existence of $G^M$ type subgraphs in an ascending order of size starting with the manufacturer $m_1$ and separates them from the graph iteratively until no more $G^M$ type subgraphs can be found. Figure 6 provides an example. The algorithm first checks all the subsets with two manufacturers including $m_1$ such as $\{m_1, m_2\}$, $\{m_1, m_3\}$, and so on to see whether any of the subsets is collectively linked to only one supplier. In this example, there is no such subset with two manufacturers that are linked to only one supplier. So, the algorithm moves on to the subsets with three manufacturers in a lexicographic order to check if any of the subsets is collectively linked with less than three suppliers. The algorithm finds that the subset of three manufacturers $\{m_2, m_3, m_4\}$ is collectively linked to only two suppliers $\{s_3, s_4\}$. Thus, the algorithm removes the manufacturers $\{m_2, m_3, m_4\}$, the suppliers $\{s_3, s_4\}$, and the links among them from the graph and continues to examine the remaining graph in a similar fashion. The algorithm checks all possible subgraphs for the existence of $G^M$ type subgraphs $|M|$ times while changing the starting manufacturer to $m_2$, $m_3$, and so on. At the end of each iteration, the algorithm finds the surplus maximizing matching and, later, chooses the highest surplus creating decomposition. The algorithm labels corresponding subgraphs as $(G, \nu)^s_M$ where $s$ is the subindex of the critical soft buyer in that subgraph. So, in the figure, the subgraph with the manufacturers $\{m_2, m_3, m_4\}$ and the suppliers $\{s_3, s_4\}$ that the algorithm identified is the only type $(G, \nu)^s_M$ subgraph in this example and is labeled as $G^M_4$ since the critical soft buyer in the subgraph is $m_4$.

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A detailed description of this algorithm can be found in the appendix.
The remaining disconnected subgraphs are necessarily of the type of subnetwork structures. The algorithm labels corresponding subgraph as \((G_v)\).

Figure 6(b) shows the subgraph \(G_4^M\) and the remaining graph after removing \(G_4^M\).

Once the algorithm removes the \(G^M\) type of subgraphs, it repeats the same process on the remaining graph in order to identify and remove the \(G^S\) type of subgraphs. In Figure 6, the algorithm first identifies the supplier set \(\{s_1, s_2\}\), which is collectively linked to the manufacturer \(m_1\) only and removes the subgraph with suppliers \(\{s_1, s_2\}\), the manufacturer \(m_1\), and the links among them from the graph. The algorithm labels corresponding subgraph as \((G_v)\). Figure 6(c) shows the two subgraphs, \(G_4^M\) and \((G_v)\) the algorithm identifies so far and the remaining graph after the the subgraphs are removed. The remaining disconnected subgraphs are necessarily \(G^E\) type. Notice that in Figure 6(c), after the \(G_4^M\) and \(G^S\) type of components are removed, the only remaining graph has the manufacturer set \(\{m_5, m_6\}\) and the supplier set \(\{s_5, s_6\}\), together with their relevant links. The shape of the this remaining graph will be similar to the one in Figure 2(c).

For the \(G^E\) type of subnetwork structures, the algorithm uses the information about the manufacturer valuations to decompose the set of \(G^E\) type subgraphs further into smaller subgraphs as follows. First, the algorithm finds all moderate manufacturers and remove all links between the following nodes: (i) suppliers who are connected to the moderate manufacturer with the highest valuation and (ii) manufacturer who has valuations lower than the highest valued moderate manufacturer. Then, the algorithm removes the links between suppliers who are connected to the moderate manufacturer with the second highest valuation and manufacturers who have valuations...
lower than the second highest valued moderate manufacturer. The algorithm continues removing links in this fashion until there are no moderate manufacturers. At the end of this process, we may end up having more than one subgraph. The algorithm labels these subgraphs as \((G, v)^E_k\) where \(k\) is the subindex of the manufacturer with the lowest valuation in that subgraph. In Figure 6(d), the algorithm identifies that both \(m_5\) and \(m_6\) are moderate manufacturers (see Figure 2(c) for discussions for a similar case). Since the moderate manufacturer with the highest valuation is \(m_5\) in this example, the algorithm removes the link between \(s_5\) (i.e., supplier who is connected to the highest valuation moderate manufacturer) and \(m_6\) (i.e., manufacturer who has valuation lower than the highest valued moderate manufacturer). After removing the link between \(s_5\) and \(m_6\), we end up having two independent subgraphs and the algorithm labels these subgraphs as \(G_5^E\) and \(G_6^E\) since the moderate manufacturers in the components are \(m_5\) and \(m_6\), respectively.

Notice that at the end of the algorithm the manufacturer with the lowest valuation in the subnetwork \((G, v)^E_k\) has to be the only moderate manufacturer in that subnetwork. Thus, the number of moderate manufacturers determines the number of subnetworks that are of type \((G, v)^E_k\). Intuitively, for \((G, v)^E_k\) type of subnetworks, the algorithm removes all the redundant links from the suppliers’ perspectives since being connected to a moderate manufacturer limits the minimum earning of a supplier. Suppliers that are connected to moderate manufacturers will never consider trading with the manufacturers with lower valuations, which implies the underlying graph structure of the supply chain can be decomposed further into smaller subgraphs.

### 4.3 Equilibrium Prices in General Supply Chain Networks

We are now ready to show the subgame perfect Nash equilibrium for a general supply chain network which can be decomposed into three types of subnetworks using the decomposition algorithm presented in Section 4.2. The following proposition specifies the subgame perfect Nash equilibrium for each of the three types of subnetwork in a general supply chain network.

**Proposition 3.** Let \((G, v)\) be a supply chain network. Then, there exists a unique (in terms of payoffs) subgame perfect Nash equilibrium such that a supplier trades with a manufacturer \(m_i\) with valuation \(v_i\) in a subnetwork \((\tilde{G}, \tilde{v})\) at price

\[
\tilde{p}_i = \begin{cases} 
0, & \text{if } (\tilde{G}, \tilde{v}) \text{ is of type } (G, v)^S; \\
\frac{\delta}{1+\delta} \tilde{v}_k, & \text{if } (\tilde{G}, \tilde{v}) \text{ is of type } (G, v)^E_k; \\
\tilde{v}_s, & \text{if } (\tilde{G}, \tilde{v}) \text{ is of type } (G, v)^M_s.
\end{cases}
\]

where \(\tilde{v}_k\) and \(\tilde{v}_s\) are the moderate and critical soft manufacturers in the subnetwork \((\tilde{G}, \tilde{v})\) when \((\tilde{G}, \tilde{v})\) is of type \((G, v)^E_k\) and \((G, v)^M_s\), respectively. The equilibrium payoff for the supplier is \(\tilde{p}_i\),
and the equilibrium payoff for the manufacturer \( m_i \) is \( v_i - \tilde{p}_i \).

The proof of Proposition 3 is by induction on the number of firms on both sides of a supply chain network. First, the small supply chain results in Propositions 1 and 2 in Section 3 are special cases of Proposition 3. Let’s check Proposition 3 on the networks in Propositions 1 and 2. The network in part (i) of Proposition 1 with one supplier linked with one manufacturer is a \((G, \mathbf{v})^E_k\) type with \( \tilde{v}_k = v \), so the equilibrium price is \( \delta \frac{1}{\frac{1}{\delta^2}} \). The network in part (ii) with two suppliers linked with one manufacturer is a \((G, \mathbf{v})^S\) type, so the equilibrium price is 0. The networks in part (iii) and (iv) with one supplier linked with two manufacturers are \((G, \mathbf{v})^M_s\) type with \( \tilde{v}_s = v \) and \( \tilde{v}_s = v_2 \), respectively, so the equilibrium prices are \( v \) and \( v_2 \) respectively. Therefore, Proposition 3 is true for the supply chain networks that have at most three firms. Now consider the networks in Proposition 2. The networks in part (i) of Proposition 2 are all \((G, \mathbf{v})^E_k\) type with \( \tilde{v}_k = v \), so the equilibrium price is \( \delta \frac{1}{\frac{1}{\delta^2}} v \). The networks in part (ii) are also \((G, \mathbf{v})^E_k\) type, but with \( \tilde{v}_k = v_2 \), so the equilibrium price is \( \delta \frac{1}{\frac{1}{\delta^2}} v_2 \). The network in part (iii) can be decomposed into two independent \((G, \mathbf{v})^E_k\) type subnetworks with \( \tilde{v}_k = v_1 \) and \( \tilde{v}_k = v_2 \), respectively, so the equilibrium prices are \( \delta \frac{1}{\frac{1}{\delta^2}} v_1 \) and \( \delta \frac{1}{\frac{1}{\delta^2}} v_2 \), respectively. Therefore, Proposition 3 is true for the supply chain networks that have at most two suppliers and two manufacturers. Then, we assume that Proposition 3 is true for supply chain networks that have at most \( n - 1 \) suppliers and \( n - 1 \) manufacturers, and show that the proposition also holds for supply chain networks that have at most \( n \) suppliers and \( n \) supply chain networks.

In a type \((G, \mathbf{v})^S\) network, we have more supply (suppliers) than demand (manufacturers). The supply and demand balance in this case favors the manufacturers and presses the prices down to the suppliers’ reservation values, which are homogenous zero. Thus, the suppliers capture zero surplus from the trades while the manufacturers extract all the surplus. However, in a type \((G, \mathbf{v})^M_s\) network, we have more demand (manufacturers) than supply (suppliers). The supply and demand balance in this case favors the suppliers and pushes the prices up to the valuation of the critical soft manufacturer, \( \tilde{v}_s \). As a result, in a type \((G, \mathbf{v})^M_s\) network, valuation heterogeneity among the manufacturers limits the un-favoring supply and demand balance situation. With the protection of valuation heterogeneity, the manufacturers are able to retain some surplus in a disadvantageous market for them. Finally, in a type \((G, \mathbf{v})^E_k\) network, there are equal numbers of suppliers and manufacturers, that is, supply and demand are perfectly balanced. So, all manufacturers will be able to trade. The equilibrium prices of trades are determined according to the moderate manufacturer’s valuation \( \tilde{v}_k \), which is the lowest among valuations of all manufacturers. Hence, valuation heterogeneity protects the manufacturers and allows them to collect some surplus in a type \((G, \mathbf{v})^E_k\) network too. Note that in \((G, \mathbf{v})^M_s\) network and \((G, \mathbf{v})^E_k\) network, the equilibrium prices depend on the exact valuations of the critical soft manufacturer and the moderate manufacturer respectively.
Therefore, there could be more prices or price dispersion than the cases where manufacturers have homogenous valuation.

In Figure (7), we apply the Proposition 3 to the complex supply chain network that is considered in Figure (6). By applying the decomposition algorithm on the network, we have identified a $G^S$, a $G^M_4$, a $G^E_5$ and a $G^E_6$ subnetwork. Then, according to Proposition 3, we can determine the equilibrium prices in each of these subnetwork: The price in $G^S$ is 0, in $G^M_4$ is $v_4$, in $G^E_5$ is $\frac{\delta}{1+\delta}v_5$ and in $G^E_6$ is $\frac{\delta}{1+\delta}v_6$.

5 Competitive and Efficient Supply Chain Networks

In this section, we investigate what types of supply chain network are competitive and/or efficient. A competitive supply chain network is essentially free from any possible rigidities and offers competitive prices, which will be defined soon below. An efficient supply chain ensures that manufacturers who trade in equilibrium are the ones with the highest valuations so that the surplus of the trades is maximized. Note that when the manufacturers have homogenous valuations, any supply chain network is efficient, that is, efficiency is not a concern at all. However, with heterogeneous valuations, not all supply chain networks can be efficient. It is worth noting that a complete network where all manufacturers and suppliers are linked with each other is both competitive and efficient.

5.1 Competitive Supply Chain Networks

To define the competitive supply chain network in our setting, we focus on a basic property of the competitive equilibrium: the equilibrium price is solely determined by the aggregate demand-supply balance in the network and other factors, such as network structure, should not restrict the equilibrium price or competition. In particular, we define the “competitive supply chain network” as follows.
Definition 5. A supply chain network \((G, \mathbf{v})\) is competitive, if

(i) there are more suppliers than manufacturers \((|M| < |S|)\) in the network, then the equilibrium price is zero for all trades;

(ii) there are less suppliers than manufacturers \((|M| > |S|)\) in the network, then the equilibrium price is equal to the \((|S| + 1)\)th highest manufacturer valuation for all trades;

(iii) the number of suppliers is equal to the number of manufacturers \((|M| = |S|)\) in the network, then the equilibrium price is equal to \(\frac{\delta}{1+\delta} \mathbf{v}\) for all trades.

Note that in a competitive network, all trades must happen at the same one price in equilibrium according to the above definition. Not every network structure allows a supply chain network to be competitive. For example, consider the supply chain network in Figure 8 where three manufacturers \((m_1, m_2, \text{ and } m_3\) with valuations \(v_1, v_2, \text{ and } v_3, \text{ respectively})\) and two suppliers \((s_1 \text{ and } s_2)\). Because there are more manufacturers than suppliers, to be competitive, all trades in this supply chain network must happen at the price of \(v_3\) according to the definition. The network on the left in Figure 8 is a competitive enough to generate such a price for all trades, whereas the one on the right is not competitive enough to make it happen. In the latter case, notice that the network can be decomposed into a \(G^M_3\) subnetwork and a \(G^E_2\) subnetwork. We know immediately there will be two different prices in the equilibrium. While manufacturer \(m_1\) will trade with supplier \(s_1\) at price \(v_3\) in the \(G^M_3\) subnetwork, manufacturer \(m_2\) will bargain with the supplier \(s_2\) in the \(G^E_2\) subnetwork independently and will pay \(s_2\) only \(\frac{\delta}{1+\delta} v_2\), which is less than the competitive price \(v_3\). The next proposition characterizes what types of supply chain network are competitive.

Proposition 4. A supply chain network \((G, \mathbf{v})\) is competitive if and only if (i) \((G, \mathbf{v})\) decomposes into subnetworks of a unique type, (ii) the only moderate manufacturer is the manufacturer with the
lowest valuation among all manufacturers when \((G,v)\) decomposes into type \((G,v)^E_k\) subnetworks, (iii) a manufacturer \(m_i\) is a soft manufacturer if and only if \(v_i \leq v^*\), where \(v^*\) is the \((|S| + 1)th\) highest valuation when \((G,v)\) decomposes into type \((G,v)^M_s\) subnetworks.

As we mentioned above, to be competitive, all trades in a supply chain network must happen at the same one competitive price that is defined purely based on its supply and demand balance. Proposition 3 has shown that the equilibrium prices in different types of subnetworks are different. Therefore, a necessary condition for a supply chain network to be competitive (part (i) of the above proposition) is that it only can be decomposed into a unique type of subnetworks so that the equilibrium prices in its subnetworks can possibly all be the same as the competitive price. This condition is also sufficient when manufacturers have homogeneous valuations (Corominas-Bosh 2004). However, it is not sufficient when manufacturers have different valuations. It is well-known that heterogeneity or differentiation of buyers could hinder competition in a network or economy. Therefore, the conditions for a supply chain network to be competitive should be more restrictive with heterogeneous manufacturer valuations. This is reflected by the fact that definitions of network types are more restrictive with heterogeneous manufacturer valuations than with homogenous manufacturer valuations. Note that we have type \((G,v)^E_k\) and type \((G,v)^M_s\) with heterogeneous manufacturer valuations, which are more restrictive than their counterparts type \((G,v)^E\) and type \((G,v)^M\) with homogenous manufacturer valuations (because of the subscripts \(k\) and \(s\)). Parts (ii) and (iii) of the proposition specify additional conditions associated with moderate manufacturers in type \((G,v)^E_k\) subnetworks and soft manufacturers in type \((G,v)^M_s\) subnetworks.

Figure 9 provides two examples to illustrate the impact manufacturer valuation heterogeneity on the competitiveness of the networks. With homogeneous manufacturer valuation \(v\), both networks (i) and (ii) in Figure 9 are competitive because they all decompose into subnetworks of a unique type (as shown in column (a) in Figure 9, network (i) decomposes to a type \(G^E\) network and network (ii) decomposes to a type \(G^M\) network). The equilibrium price in network (i) is \(\frac{\delta}{1+\delta}v\) and the equilibrium price in network (ii) is \(v\), which both are competitive prices. With heterogeneous valuations, the decompositions of the two networks are different and the definitions of types of subnetworks are more restrictive. As shown in column (b) in Figure 9, network (i) decomposes into two different types of subnetworks: one type \((G,v)^E_1\) and one type \((G,v)^E_2\), and network (ii) decomposes into two different types of subnetworks: one type \((G,v)^M_3\) and one type \((G,v)^M_5\). As a result, both networks are not competitive according to Proposition 4. In fact, according to Proposition 3, the equilibrium of network (i) is that \(m_1\) trades with \(s_1\) at price \(\frac{\delta}{1+\delta}v_1\) and \(m_2\) trades with \(s_2\) at price \(\frac{\delta}{1+\delta}v_2\), while the competitive price is \(\frac{\delta}{1+\delta}v_2\); the equilibrium of network (ii) is that \(m_1\) trades with \(s_2\) at price \(v_1\) and \(m_2\) trades with \(s_1\) at price \(v_2\), while the competitive price is \(v_3\).
As the above discussions indicate, valuation heterogeneity could create more frictions in a supply chain network. Thus, valuation heterogeneity reduces the set of networks that are competitive. This especially can be the case when the number of the manufacturers in a supply chain network is large and the network structure is far from being complete. To make a supply chain network more competitive, we must build new links into the supply chain network. Next, we consider the impact of adding one new link on the competitiveness of a supply chain network. Recall that a competitive supply chain network has only one price in equilibrium. Thus, the price dispersion of the equilibrium prices (i.e., number of prices occur in the equilibrium) can serve as a measure of competitiveness of a supply chain network. Let $\rho(G, v)$ denote the number of equilibrium prices in the supply chain network $(G, v)$. For the ease of exposition, we drop $v$ from the competitiveness measure and simply write $\rho(G)$ whenever understood.

**Definition 6.** A supply chain network $(G, v)$ becomes more competitive after adding a link $ij \notin L$ between manufacturer $m_i \in M$ and supplier $s_j \in S$ in $G$ if and only if $\rho(G + ij) < \rho(G)$, where $G + ij$ is the graph created by adding the link $ij$ to $G$.

First, we identify cases that adding a new link does not improve the competitiveness of a supply chain network.

**Proposition 5.** Consider a supply chain network $(G, v)$. Let $G + ij$ be the graph created by adding a link $ij \notin L$ between manufacturer $m_i \in M$ and supplier $s_j \in S$ in $G$. Adding the link $ij$ does not
improve the competitiveness of the supply chain network, that is, $\rho(G + ij) = \rho(G)$, if

(i) manufacturer $m_i$ and supplier $s_j$ belong to the same subnetwork;

(ii) manufacturer $m_i$ belongs to a subnetwork of type $G^S$;

(iii) supplier $s_j$ belongs to a subnetwork of type $G^M$.

To improve competitiveness, adding a link must reduce the number of different equilibrium prices in a network. According to Proposition 3, reducing the number of different equilibrium prices is equivalent to reducing the number of different types of subnetworks identified by the decomposition algorithm. Part (i) of the above proposition indicates that adding a link that connects a manufacturer and supplier within the same subnetwork will not change the subnetworks identified by the decomposition algorithm, thereby reducing the number equilibrium prices and improving the competitiveness of the supply chain network. Parts (ii) and (iii) of the above proposition jointly suggest that adding a link connecting to a firm (either a supplier or a manufacturer) who is on the short side of a subnetwork will not improve the competitiveness of a supply chain network. In part (ii), if a manufacturer belongs to a $G^S$ subnetwork with more suppliers than manufacturers, the manufacturer is already on the advantageous side of the subnetwork and will get all the surplus from the trade. This manufacturer does not need any more links to improve his position and surplus. Therefore, any new link connecting to this manufacturer is redundant to her and will be ignored by her. She will keep bargaining in the same subnetwork as she used to before adding the link. As a result, the competitiveness of the supply chain cannot be improved by adding a link connecting to such a manufacturer. For similar reasons, in part (iii), adding any link connecting to a supplier belongs to a $G^M$ subnetwork with more manufacturers than suppliers will not improve the competitiveness of the supply chain network either.

The above discussions imply that to improve the competitiveness of a network, it is necessary to add a link that connects firms that are on the long (disadvantageous) sides of two different subnetworks. These firms would appreciate and utilize the additional link, which possibly lets them mitigate their disadvantages by reaching out to more potential trading partners outside of their current subnetworks. So, competition will spread across the two local subnetworks through the added link and the two subnetworks will merge into one larger subnetwork. As a result, the number of subnetworks as well as the number of equilibrium prices will reduce and the competitiveness of the supply chain network will improve. The following proposition identifies some links that can achieve this goal.

**Proposition 6.** Consider a supply chain network $(G, v)$. Let $G + ij$ be the graph created by adding
a link $ij \notin L$ between manufacturer $m_i \in M$ and supplier $s_j \in S$ in $G$. Adding the link $ij$ improves the competitiveness of the supply chain network, that is, $\rho(G + ij) < \rho(G)$, if

(i) $m_i$ and $s_j$ belong to subnetworks of type $G^E_k$ and $G^E_1$, respectively, and

\begin{enumerate}
  \item $m_i = m_k$, $m_l \in N_{G+ij}(s_j)$, and $|N_{G+ij}(m_l)| > 1$, or
  \item $m_i = m_k$, $m_l \in N_{G+ij}(s_j)$, $|N_{G+ij}(m_l)| = 1$, and $v_k > v_l$, or
  \item $m_i = m_k$, $m_l \notin N_{G+ij}(s_j)$ and $m_k$ does not have the lowest valuation in $N_{G+ij}(s_j)$, or
  \item $m_i \neq m_k$, $m_l \in N_{G+ij}(s_j)$, $|N_{G+ij}(m_l)| > 1$, and $m_l$ does not have the lowest valuation in $N_{G+ij}(s_j)$;
\end{enumerate}

(ii) $m_i$ belongs to a subnetwork $G_i$, which is of type $G^M_s$ with one soft manufacturer, and

\begin{enumerate}
  \item $s_j$ belongs to a subnetwork of type $G^E_k$, or
  \item $s_j$ belongs to a subnetwork $G_j$ which is of type $G^S$, and $|S_i + S_j| = |M_i + M_j|$ (i.e., total number of suppliers in $G_i + G_j$ is equal to total number of manufacturers in $G_i + G_j$).
\end{enumerate}

(iii) $m_i$ and $s_j$ belong to subnetworks of type $G^E_k$ and $G^S$, respectively.

Part (i) of the above proposition considers a new link that connects a manufacturer in a $G^E_k$ subnetwork whose moderate manufacturer is $m_k$ to a supplier in a $G^E_1$ subnetwork whose moderate manufacturer is $m_l$. Recall that the moderate manufacturer in a $G^E$ subnetwork has either the lowest valuation or only one link, and a $G^E$ subnetwork has only one moderate manufacturer. To reduce the number of subnetworks and improve the competitiveness of the supply chain network, adding a link in part (i) must merge the two $G^E$ type subnetworks into one and make one of the two moderate manufacturers, $m_k$ and $m_l$ to be non-moderate. Parts (ia), (ib) and (ic) consider adding links that connect the moderate manufacturer $m_k$ in the $G^E_k$ subnetwork to a supplier $s_j$ in the $G^E_1$ subnetwork.

Part (ia) says that adding a link connecting the moderate manufacturer $m_k$ to a supplier $s_j$ which connects to the moderate manufacturer $m_l$ with more than one link in the $G^E_1$ subnetwork improves the competitiveness of the supply chain. With this newly added link, the two moderate manufacturers in two different $G^E$ subnetworks, $m_k$ and $m_l$ are now both connected with the supplier $s_j$. Therefore, one of these two manufacturers will not be a moderate manufacturer anymore. In other words, adding such a link will merge the two subnetworks into one subnetwork with only one moderate manufacturer, thereby improving the competitiveness of the network. Part (ib) says that adding a link connecting the moderate manufacturer $m_k$ to a supplier $s_j$ which connects to
the moderate manufacturer \( m_l \) with only one link in the \( G_l^E \) subnetwork, would improve the competitiveness of the supply chain, too, if manufacturer \( m_k \) has higher valuation than manufacturer \( m_l \) (i.e., \( v_k > v_l \)). In this case, the two subnetworks will merge and manufacturer \( m_k \) will not be a moderate manufacturer anymore because of \( v_k > v_l \). Part (ic) considers adding a link connecting the moderate manufacturer \( m_k \) to a supplier \( s_j \), which does not connect to the moderate manufacturer \( m_l \) in her \( G_l^E \) subnetwork. If manufacturer \( m_k \) does not have the lowest valuation among all his neighbors after the link is added, he will not be a moderate manufacturer anymore. So, the two subnetwork will merge and the only remaining moderate manufacturer will be manufacturer \( m_l \).

Part (id) is opposite to part (ic). It says that adding a link to connect a non-moderate manufacturer in the \( G_k^E \) subnetwork to the supplier \( s_j \) that is linked to the moderate manufacturer \( m_l \) in her \( G_l^E \) subnetwork will improve competitiveness of the supply chain if the moderate manufacturer \( m_l \) does not have the lowest valuation among his neighbors after the link is added. In this case, with the new link, manufacturer \( m_l \) will not be a moderate manufacturer anymore and the two subnetwork will merge into one. As a result, the competitiveness of the supply chain network improves. In summary, one possible way for a new link to improve the competitiveness of a supply chain network is that at least one manufacturer in the network, which normally is a moderate manufacturer in a \( G^E \) subnetwork, must be able to utilize this link to improve her bargaining position in the network so that a lower equilibrium price can emerge.

Part (ii) of Proposition 6 considers a new link that connects a manufacturer in a \( G_s^M \) subnetwork, which has more manufacturers than suppliers, to a supplier in another subnetwork. Part (iia) indicates that adding a link that connects a manufacturer in a \( G_s^M \) subnetwork to a supplier in a \( G_k^E \) subnetwork would improve the competitiveness of the supply chain network. Both the supplier and the manufacturer on this link would potentially benefit from the link. The supplier in the \( G_k^E \) subnetwork, where he has no advantage because of balanced supply and demand, would benefit from connecting to the \( G_s^M \) subnetwork with more manufacturers. The manufacturer in the \( G_s^M \) subnetwork could benefit from higher possibility of obtaining a good by connecting to more suppliers in the \( G_k^E \) network. Thus, this new link will be utilized by the two firms and the two subnetworks will merge into a new \( G^M \) type subnetwork. Part (iib) suggests that adding a link that connects a manufacturer in a \( G_s^M \) subnetwork to a supplier in a \( G^S \) subnetwork, which has more suppliers than manufacturers, would improve the competitiveness of the supply chain network if the two subnetworks merge into a \( G^E \) type subnetwork. In this case, both the manufacturer in the \( G_s^M \) subnetwork and the supplier in the \( G^S \) subnetwork suffer from unfavorable mismatch between supply and demand, i.e., they are all on the long sides of their subnetworks. If the two subnetworks merge to a \( G^E \) type subnetwork (i.e., \(|S_i + S_j| = |M_i + M_j|\)) which has the same number of manufacturers and suppliers, the new link would lead to a better match between supply
and demand for both firms. So, both firms would appreciate and utilize the link, and the two subnetworks would merge.

Part (iii), which is parallel to Part (iia), says that adding a link that connects a manufacturer in a $G^E_k$ subnetwork to a supplier in a $G^S$ subnetwork would improve the competitiveness of the supply chain network because both the manufacturer and the supplier can benefit from the link as they would in the case of Part (iia).

5.2 Efficient Supply Chain Networks

Network structure not only impacts the equilibrium prices but also determines the set of feasible trading patterns or allocation of goods in a supply chain network. In a supply chain network, it is possible that manufacturers who have the highest valuations are not necessarily able to trade in equilibrium due to the restrictions of the network structure. This inefficiency will lead to a loss of economic surplus in the supply chain network. In order to examine the welfare properties of supply chain networks, we define a supply chain network $(G, v)$ as “efficient” if only the manufacturers with the highest valuations are able to trade in the equilibrium. Note that with homogeneous manufacturer valuations, this is not a concern and all supply chain networks are efficient.

Proposition 7. A supply chain network $(G, v)$ is efficient if and only if after decomposition, either there are no $G^M_s$ subnetworks or all soft manufacturers in $G^M_s$ subnetworks have valuations less than or equal to the $(|S| + 1)$th highest manufacturer valuation.

Recall that a soft manufacturer is defined for type $G^M_s$ network in which there are more manufacturers than suppliers. If a supply chain network can be decomposed into only type $G^S$ and type $G^E_k$ subnetworks, there are no soft manufactures. In type $G^S$ or type $G^E_k$ subnetworks, there is enough supply (i.e., suppliers) to guarantee all manufacturers in the networks to be able to trade in equilibrium. So, these two types of subnetworks always provide efficient allocations. Therefore, the original supply chain network is efficient. However, if the decomposition of a supply chain network includes type $G^M_s$ subnetworks, there is a supply shortage (i.e., less suppliers than manufacturers) in these subnetworks so that soft manufacturers who are not well connected (i.e., with only one link) will not be not able to trade. Then, allocation efficiency becomes a concern. To ensure that the manufacturers with the highest valuations in a type $G^M_s$ subnetwork are able to trade, they should be well connected with more than one link, that is, not be the soft manufacturers in the type $G^M_s$ subnetworks.

Figure 10 provides examples of inefficient and efficient supply chain networks. The network on the left is inefficient since $m_2$ cannot trade in the equilibrium. If we apply the Proposition 7 to this example, the supply chain network is decomposed into a $G^M_2$ subnetwork and a $G^E_3$ subnetwork.
(a) Inefficient supply chain network  (b) Efficient supply chain network

Figure 10: Inefficient vs. efficient supply chain networks.

The soft manufacturer in the $G^M_2$ subnetwork is manufacturer $m_2$ whose valuation $v_2$ is higher than the third ($|S|+1 = 3$) highest manufacturer valuation, which is $v_3$. In this example, the inefficiency is caused by the fact that manufacturer $m_2$ is not well connected (due to only one link) and has to compete with manufacturer $m_1$ to trade with supplier $s_1$, while manufacturer $m_3$ with lower valuation is well connected and can be guaranteed to trade. If we switch the positions of $m_2$ and $m_3$, we have the network on the right which is an efficient network because in the equilibrium, $m_1$ and $m_2$ who have the highest valuations in the network will trade while $m_3$ will not. If we apply the Proposition 7 to this example, the supply chain network is decomposed into a $G^M_3$ subnetwork and a $G^E_2$ subnetwork. The soft manufacturer in the $G^M_3$ subnetwork is manufacturer $m_3$, whose valuation $v_3$ is indeed the third highest manufacturer valuation.

**Proposition 8.** An efficient supply chain network may not be competitive whereas a competitive supply chain network must be efficient.

Proposition 8 implies that the conditions for a competitive network are more restrictive than conditions for an efficient network. To be competitive, as part (i) Proposition 4 suggests, a supply chain network must decompose into subnetworks of a unique type so that all trades take place at the same price. If it only decomposes to $G^E_k$ or $G^S$ types of subnetworks, the supply chain network must be efficient too because all manufacturers in these two types of subnetworks will be able to trade. If the network only decomposes to $G^M_s$ type subnetworks, the conditions specified in part (iii) of Proposition 4 will ensure only manufacturers with the highest valuations trade. However, an efficient network, where manufacturers with the highest valuations trade, may not be competitive because these manufacturers may trade at different prices in equilibrium. Therefore, there could exist a divergence between the efficiency of a network, which is a supply chain network level concern, and the competitiveness of a network, which is a firm level concern. Figure 8 provides such an example that highlights this divergence between efficiency and competitiveness. The network on the right side of Figure 8 is efficient (because $m_1$ and $m_2$ are able to trade and total surplus is
maximized) but is not competitive (see the discussion in Section 5.1 for details).

Next, we consider the impact of adding a new link on the efficiency of a supply chain network. Because efficiency is about the total surplus, we use the total surplus in equilibrium as a measure of the degree of efficiency of a supply chain network. Let $\mu(G)$ be the total surplus created in the equilibrium for the supply chain network $(G, v)$.

**Definition 7.** A supply chain network $(G, v)$ becomes more efficient after adding a link $ij \notin L$ between manufacturer $m_i \in M$ and supplier $s_j \in S$ in $G$ if and only if $\mu(G + ij) > \mu(G)$, where $G + ij$ is the graph created by adding the link $ij$ to $G$.

**Proposition 9.** Consider a supply chain network $(G, v)$. Let $G + ij$ be the graph created by adding a link $ij \notin L$ between manufacturer $m_i \in M$ and supplier $s_j \in S$ in $G$.

1. **Adding the link $ij$ does not improve efficiency,** that is $\mu(G) = \mu(G + ij)$, if
   
   (a) $m_i$ and $s_j$ belong to the same subnetwork;
   
   (b) both $m_i$ and $s_j$ belongs to a subnetwork of type $G^S$ or $G^E_k$;
   
   (c) $s_j$ belongs to a subnetwork of type $G^M_s$.

2. **Adding the link $ij$ improves efficiency,** that is $\mu(G) < \mu(G + ij)$, if
   
   (a) $m_i$ belongs to a subnetwork of type $G^M_s$ with only one soft manufacturer, $s_j$ belongs to a subnetwork of type $G^E_k$, and $v_s > v_k$;
   
   (b) $m_i$ belongs to $G_i$, which is of type $G^M_s$ with only one soft manufacturer, $s_j$ belongs to $G_j$, which is of type $G^S$, and $|S_i + S_j| = |M_i + M_j|$.
   
   (c) $m_i$ belongs to a subnetwork of type $G^M_s$ with multiple soft manufacturers, $s_j$ belongs to a subnetwork of type $G^E_k$, and $v_i > v_k$.

To improve the efficiency of the supply chain network, adding a new link must lead to one or both of the following two changes in the network: (1) the total number of trades in the network increases so that more manufacturers will get a good; (2) the trade matching between suppliers and manufacturers changes in such a way that the manufacturers with higher valuations get the goods. First, Part (i) of the above proposition identifies what types of link do not lead to any one of the above two changes when added, therefore not improving the efficiency of the supply chain network. Part (ia) is straightforward and indicates that adding a link that connects a manufacturer and a supplier within the same subnetwork will not change the allocation of goods and the efficiency of the network. Parts (ib) identifies cases where all manufacturers in a subnetwork have been able to trade
already before adding the new link. Hence, adding the new link would not change the efficiency at all. This happens when both \( m_i \) and \( s_j \) belongs to a subnetwork of type \( G^S \) (all manufacturers are able to trade because there are more suppliers than manufacturers) or when both \( m_i \) and \( s_j \) belongs to a subnetwork of type \( G^E_k \) (all manufacturers are able to trade because there are equal number of suppliers and manufacturers). In part (ic), if a supplier belongs to a \( G^M_s \) subnetwork with more manufacturers than suppliers, then the supplier already has an advantageous bargaining position in the subnetwork and will be able to trade whether there is a new link or not. Adding a new link connecting this supplier with a manufacturer is redundant and will be ignored by the supplier. Therefore, it will not improve the efficiency of the network.

Part (ii) of the Proposition 9 identifies cases where adding a link will improve the efficiency of the supply chain network. In particular, the proposition considers a new link that connects a manufacturer in a \( G^M_s \) subnetwork, which has more manufacturers than suppliers, to a supplier in another subnetwork. Because there are more manufacturers than suppliers in a \( G^M_s \) subnetwork, some manufacturers will not be able to trade, which provides opportunity for improvement. Part (iia) says that adding a link that connects a manufacturer in a \( G^M_s \) subnetwork with only one soft manufacturer to a supplier in a \( G^E_k \) subnetwork would improve the efficiency of the supply chain network. While the total number of trades stays the same in such a case, the additional link improves the efficiency by allowing better allocation of the goods: i.e., the manufacturer \( m_s \) who is the critical soft manufacturer of subnetwork \( G^M_s \) will receive a good instead of \( m_k \), who is the moderate manufacturer of subnetwork \( G^E_k \), because \( m_s \) has higher valuation than \( m_k \) (i.e., \( v_s > v_k \)). Part (iib) suggests that adding a link that connects a manufacturer in a \( G^M_s \) subnetwork to a supplier in a \( G^S \) subnetwork, which has more suppliers than manufacturers would improve the efficiency of the supply chain network if the two subnetworks merge into a \( G^E \) type subnetwork (if \( |S_i + S_j| = |M_i + M_j| \)) which has the same number of suppliers and manufacturers. Such a merge would lead to better match between supply and demand in the supply chain network which improves efficiency. Part (iic) of the proposition demonstrates the case where after adding a link between manufacturer \( m_i \) and supply \( s_j \), the manufacturer \( m_i \) would leave her original subnetwork and join the supplier \( s_j \)'s subnetwork that has lower valuation manufacturers. As a result, although the number of trades does not change in the supply chain network, the manufacturer \( m_i \) benefits from the additional link since it allows her to compete with lower valuation manufacturers. So, adding the link improves the efficiency since \( m_i \) will replaces supplier \( s_j \)'s subnetwork’s original moderate manufacturer \( m_k \), who has lower valuation than \( m_i \) (\( v_i > v_k \)), to trade in the new subnetwork.

**Proposition 10.** Consider a supply chain network \((G, v)\). Let \( G + ij \) be the graph created by adding a link \( ij \notin L \) between manufacturer \( m_i \in M \) and supplier \( s_j \in S \) in \( G \).
Figure 11: Adding a link improves the efficiency but not competitiveness.

(i) Adding the link $ij$ to $G$ improves the competitiveness but not efficiency when $m_i$ belongs to a subnetwork of type $G^M_s$ with only one soft manufacturer, $s_j$ belongs to a subnetwork of type $G^E_k$, and $v_s < v_k$;

(ii) Adding the link $ij$ to $G$ improves the efficiency but not competitiveness when $m_i$ belongs to a subnetwork of type $G^M_s$ with multiple soft manufacturers, $s_j$ belongs to a subnetwork of type $G^E_k$, and $v_i > v_k$.

According to our discussions above, the requirements for improving competitiveness and for improving efficiency of a network are different. Improving competitiveness requires reducing the number of equilibrium prices or the number of subnetworks, while improving efficiency requires better allocation of goods. Therefore, there could exist a divergence between the effect of adding a new link into a supply chain network on the competitiveness and the efficiency of the network: While some links improve only the efficiency, other links may improve only competitiveness or both.

Part (i) of Proposition 10 identifies a case where adding a link improves competitiveness but not efficiency. In this case, the new link between manufacturer $m_i$ and supplier $s_j$ will combine a $G^M_s$ subnetwork and a $G^E_k$ subnetwork into one bigger $G^M_s$ subnetwork, which improves competitiveness as the number of subnetworks is reduced. However, the trades in the new combined subnetwork are the same as the trades in the two subnetworks before the link is added. In other words, the allocation of goods stays unchanged after the link is added. As a result, adding the new link does not improve the efficiency of the network. Part (ii) of Proposition 10 identifies an opposite case where adding a link improves efficiency but not competitiveness. In this case, according to part (iic) of Proposition 9, adding the link between manufacturer $m_i$ and supplier $s_j$ improves efficiency because manufacturer $m_i$ will join supplier $s_j$’s subnetwork and replace its original moderate manufacturer $m_k$, who has lower valuation, to trade. However, after manufacturer $m_i$ leaves its original $G^M_s$ subnetwork to join supplier $s_j$’s $G^E_k$ subnetwork, we still have two subnetworks as before. Thus, competitiveness of the network is not improved. Figure 11 provides an example of this case.
6 Conclusion

This paper studies bargaining between manufacturers and suppliers in a supply chain network where firms must have an established business relationship or “link” to bargain and trade. The network structure of a supply chain restricts the equilibrium trading pattern in the network. We model such a supply chain via a bipartite graph. Through this model, we examine how supply and demand balance, network structure, and valuation heterogeneity among manufacturers jointly influence the equilibrium prices and trading pattern of the supply chain networks.

We have shown that valuation heterogeneity among manufacturers plays an important role in interacting with supply and demand balance and network structure of a supply chain network to influence the equilibrium outcome of the supply chain network. Valuation heterogeneity among manufacturers serves as a counter force to unfavorable supply and demand imbalance to protect the manufacturers from surplus extractions by the suppliers. Therefore, whether a supplier can get a good deal in the bargaining in a supply chain network not only depends on how many manufacturers he has a link with, but also depends on the quality of the manufacturers (in terms of valuations) he has a link with.

We also found that bargaining actually takes place in smaller subnetworks in a general supply chain network as firms may selectively bargain with a subset of firms with whom they are linked. In other words, firms may ignore some of their links in the network that are not useful for them to get a better deal. We develop a decomposition algorithm to identify these subnetworks in which bargaining between firms will effectively take place in general supply chain networks. This implies that a firm should select the right set of firms on the other side of the supply chain network with whom it has a link to bargain in order to get a good deal.

For a supply chain network to be competitive, all trades must happen at the single competitive price, which is defined purely by the aggregate supply and demand balance in the network without any restriction of the network structure. For a supply chain network to be efficient, only the manufacturers with the highest valuations can trade. We demonstrate that the conditions required to make a supply chain network competitive are quite different from the ones to make the network efficient. It suggests that there could exist a divergency between competitiveness, which is a firm level concern, and efficiency, which is a network level concern, of a supply chain network. As a result, the types of links that can improve the competitiveness of a supply chain network are different from the ones that can improve the efficiency of the network. Adding a new link into a supply chain network can improve its competitiveness if the two firms connected by the link can do better in the bargaining than without the new link. They will appreciate and utilize this new link in their bargaining. Adding a new link into a supply chain network can improve its efficiency if
with the new link, more trades can happen in the network or manufactures with higher valuations can replace manufacturers with lower valuations to trade in the network.

There are several potential extensions of our research. We could consider the case where suppliers are asymmetric in terms of their costs. In combination with the manufacturer valuation heterogeneity, it is interesting to see how the asymmetries on both sides of the network impact the equilibrium outcome of the network. We can relax the assumption on unit demand and supply to consider the case where each supplier can supply more than one unit of the component and each manufacturer can buy more than one unit of the component. Another interesting extension could be the impact of uncertainty of supply and demand on the equilibrium of the network.

References


Appendix

A. Proofs

Proof of Proposition 1. Part (i) is the classical Rubinstein (1982) bargaining game with the trading price equal to $\frac{\delta}{1+\delta} v$. Parts (ii) and (iii) are similar to the Bertrand competition where two firms undercut each other’s price until they reach to their maximum willingness to pay (which is $v$). Thus, the price in (ii) and (iii) are 0 and $v$, respectively. The last part is similar to (iii) but the manufacturer $v_2$ drops out from bargaining when the price reaches to $v_2$ and the other manufacturer has no incentive to offer a higher price.

Proof of Proposition 2. Notice that strategies must specify the distribution of proposed prices and the responses of the firms not only when the supply chain graph is $G$, but also when the supply chain graph is given by any subgraph that results from $G$ after a supplier and manufacturer trades and leaves the market.

When there are only four firms in the supply chain, two possibilities can happen: (1) one pair trades and two firms get isolated or (2) one pair trades and the remaining two firms are still connected and can keep playing. The strategies followed in any of these subgraphs are simple. If firms get isolated, they automatically get zero and have no actions to choose. If a pair remains in the market, then the strategies are as in the two players alternating offers bargaining game. Notice that although the strategies below are described for a more general case, our tie breaking mechanism gives a higher priority to the manufacturer with higher valuation.


(ii) Let $p_{m_i}$ and $p_{s_j}$ be the offered prices by each firm.

Existence: The following strategies form a subgame perfect Nash equilibrium. If all firms are still in the initial graph $G$,

* manufacturers propose $p_m = \frac{\delta}{1+\delta} v_2$ and suppliers propose $p_s = \frac{1}{1+\delta} v_2$;

* $s_1$ accepts $p_{m_1}$ if $p_{m_1} \geq \frac{\delta}{1+\delta} v_2$, $m_2$ accepts $p_{s_2}$ if $p_{s_2} \leq \frac{1}{1+\delta} v_2$;

* $m_1$ accepts the minimum of the offered prices provided that $\min_j \{p_{s_j}\} \leq \frac{1}{1+\delta} v_2$;

* the strategy followed by $s_2$ when responding depends on the priorities determined by the tie breaking mechanism (which is random):

  · If the priority of $s_2$ is higher than that of $s_1$, then $s_2$ accepts the maximum of the offered prices provided that $\max_i \{p_{m_i}\} \geq \frac{\delta}{1+\delta} v_2$.

  · If the priority of $s_2$ is smaller than that of $s_1$ and $p_{m_1} \leq \frac{\delta}{1+\delta} v_2$, then $s_2$ accepts the maximum of the offered prices provided that $\max_i \{p_{m_i}\} \geq \frac{\delta}{1+\delta} v_2$.

  · If the priority of $s_2$ is smaller than that of $s_1$ and $p_{m_1} > \frac{\delta}{1+\delta} v_2$, then $s_2$ accepts $p_{m_1}$ when $p_{m_2} < \frac{\delta}{1+\delta} v_2$, otherwise he accepts $\min_i \{p_{m_i}\}$.
If there is only one pair of firms, \( m_i \) and \( s_j \), left in the supply chain, then

* \( m_i \) proposes \( p_{m_i} = \frac{\delta}{1+\delta} v_i \) and \( s_j \) proposes \( p_{s_j} = \frac{1}{1+\delta} v_i \);

* \( s_j \) accepts \( p_{m_i} \) if \( p_{m_i} \geq \frac{\delta}{1+\delta} v_i \), \( m_i \) accepts \( p_{s_j} \) if \( p_{s_j} \leq \frac{1}{1+\delta} v_i \).

**Uniqueness:** We refer the situation where the manufacturers (suppliers) propose prices first as a \( m \)-game (\( s \)-game). Call \( S_{m_1}, I_{m_1} \), the supremum and infimum of subgame perfect Nash equilibrium payoffs for manufacturers in a \( m \)-game (similarly, \( S_{s_i}, I_{s_i} \) for suppliers in a \( s \)-game), when all four firms are still in the market. By existence part, we already know that \( S_{m_1} \geq v_1 - \frac{\delta}{1+\delta} v_2 \geq I_{m_1} \) and \( S_{s_1}, S_{s_2}, S_{m_2} \geq \frac{1}{1+\delta} v_2 \geq I_{s_1}, I_{s_2}, I_{m_2} \).

First, notice that if \( s_2 \) rejects an offer from \( m_2 \), the maximum she may get in the next period is either \( \delta \frac{1}{1+\delta} v_2 \) (if \( m_1 \) traded with \( s_1 \) and left the market) or \( \delta S_{s_2} \) (if all firms are in the market). Thus, \( m_2 \) will never offer a price strictly smaller than \( \delta \max \{ \frac{1}{1+\delta} v_2, S_{s_2} \} \) since he is sure to be accepted by supplier \( s_2 \) when he asks for that amount. Thus, we must have

\[
I_{m_2} \geq v_2 - \delta \max \{ \frac{1}{1+\delta} v_2, S_{s_2} \} = v_2 - \delta S_{s_2}. \tag{1}
\]

On the other hand, we can also show that

\[
S_{s_2} \leq v_2 - \delta \min \{ \frac{1}{1+\delta} v_2, I_{m_2} \} = v_2 - \delta I_{m_2}. \tag{2}
\]

That is because, if there is agreement between \( m_1 \) and \( s_1 \) in the \( s \)-game, the minimum amount \( m_2 \) can get in the \( m \)-game is \( \delta \frac{1}{1+\delta} v_2 \); if there is no agreement in the \( s \)-game, the minimum amount \( m_2 \) can get in the \( m \)-game is \( \delta I_{m_2} \). Thus, the maximum that \( s_2 \) can collect from \( m_2 \) is less than or equal to \( v_2 - \delta \min \{ \frac{1}{1+\delta} v_2, I_{m_2} \} \). Since the highest value \( s_2 \) can get is \( v_2 - \delta I_{m_2} \), the inequality in (1) reduces to

\[
I_{m_2} \geq v_2 - \delta [v_2 - \delta I_{m_2}],
\]

which implies that \( I_{m_2} \geq \frac{1}{1+\delta} v_2 \). Because we know from the existence condition that \( I_{m_2} \leq \frac{1}{1+\delta} v_2 \), we must have \( I_{m_2} = \frac{1}{1+\delta} v_2 \). By substituting this equality into (2), we get

\[
S_{s_2} \leq v_2 - \delta I_{m_2} = v_2 - \delta \frac{1}{1+\delta} v_2 = \frac{1}{1+\delta} v_2
\]

Because \( S_{s_2} \geq \frac{1}{1+\delta} v_2 \) by the existence, it must be the case that \( S_{s_2} = \frac{1}{1+\delta} v_2 \).

Next, consider the inequalities related to \( m_1 \) and \( s_1 \). We can show the lower bound of \( m_1 \)'s payoffs is

\[
I_{m_1} \geq v_1 - \delta \min \left\{ \max \left\{ \frac{1}{1+\delta} v_1, S_{s_1} \right\}, \max \left\{ \frac{1}{1+\delta} v_2, S_{s_2} \right\} \right\} \tag{3}
\]

The first element of the \( \min \) function on the right hand side is the minimum amount that \( m_1 \) has to offer in order to ensure \( s_1 \)'s acceptance and the second element is the amount to ensure \( s_2 \)'s acceptance. Thus, \( m_1 \) will never offer a price less than the smallest of these two
amounts. Equation (3) reduces to \( I_{m_1} \geq v_1 - \frac{1}{1+\delta} v_2 \) since we know that \( S_{s_2} = \frac{1}{1+\delta} v_2 \) and \( \max\{\frac{1}{1+\delta} v_1, S_{s_1}\} > \frac{1}{1+\delta} v_2 \). Because \( v_1 - \frac{1}{1+\delta} v_2 \geq I_{m_1} \) by the existence, it must be the case that 
\( I_{m_1} = v_1 - \frac{1}{1+\delta} v_2 \).

On the other hand, the maximum amount that \( s_1 \) can collect is bounded above by

\[
S_{s_1} \leq v_1 - \delta \min\{I_{m_1}, I_{m_2}\} \tag{4}
\]

If \( s_1 \) offers a price strictly smaller than \( \min\{I_{m_1}, I_{m_2}\} \), none of the manufacturers will accept.

The reasoning is as follows. As we have established above, \( m_2 \) will not accept such an offer since \( I_{m_2} = \frac{1}{1+\delta} v_2 \). Anticipating \( m_2 \)'s behavior, \( m_1 \) would not accept such an offer either since by rejecting the offer, \( m_1 \) can get at least \( \delta I_{m_1} \). Since we have already established that 
\( I_{m_1} = v_1 - \frac{1}{1+\delta} v_2 \geq \frac{1}{1+\delta} v_2 = I_{m_2} \), we must have

\[
S_{s_1} \leq v_1 - \delta I_{m_2} = \frac{1}{1+\delta} v_2,
\]

which together with the existence condition implies that \( S_{s_1} = \frac{1}{1+\delta} v_2 \).

Until now, we have established the values only for \( I_{m_1}, I_{m_2}, S_{s_1}, \) and \( S_{s_2} \). We omit the rest of the proof since the remaining values can be determined by using an approach similar to the one above.

(iii) As before, let \( p_m \), and \( p_s \) be the offered prices by each firm.

Existence: The following strategies form a subgame perfect Nash equilibrium. If all firms are still in the initial graph \( G \),

* \( m_1 \) proposes \( p_{m_1} = \frac{\delta}{1+\delta} v_1 \), \( m_2 \) proposes \( p_{m_2} = \frac{\delta}{1+\delta} v_2 \), \( s_1 \) proposes \( p_{s_1} = \frac{1}{1+\delta} v_1 \) and \( s_2 \) proposes \( p_{s_2} = \frac{1}{1+\delta} v_2 \).
* \( s_2 \) accepts \( p_{m_2} \) if \( p_{m_2} \geq \frac{\delta}{1+\delta} v_2 \), \( m_1 \) accepts \( p_{s_1} \) if \( p_{s_1} \leq \frac{1}{1+\delta} v_1 \).
* the strategy followed by \( s_1 \) when responding depends on the priorities of the suppliers, which is determined by the tie breaking mechanism (which is random).

  · If the priority of \( s_1 \) is higher than that of \( s_2 \), then \( s_1 \) accepts the maximum of the offered prices provided that \( \max_i\{p_{m_i}\} \geq \frac{\delta}{1+\delta} v_1 \).
  · If the priority of \( s_1 \) is smaller than that of \( s_2 \) and \( p_{m_2} \leq \frac{\delta}{1+\delta} v_1 \), then \( s_1 \) accepts the maximum of the offered prices provided that \( \max_i\{p_{m_i}\} \geq \frac{\delta}{1+\delta} v_1 \).
  · If the priority of \( s_1 \) is smaller than that of \( s_2 \) and \( p_{m_2} > \frac{\delta}{1+\delta} v_1 \), then \( s_1 \) accepts \( p_{m_2} \) when \( p_{m_1} < \frac{\delta}{1+\delta} v_1 \), otherwise he accepts \( \min_i\{p_{m_i}\} \).

If there is only one pair of firms, \( m_i \) and \( s_j \), left in the supply chain, then
* \( m_i \) proposes \( p_{m_i} = \frac{\delta}{1 + \delta} v_i \) and \( s_j \) proposes \( p_{s_j} = \frac{1}{1 + \delta} v_i \);
* \( s_j \) accepts \( p_{m_i} \) if \( p_{m_i} \geq \frac{\delta}{1 + \delta} v_i \), \( m_i \) accepts \( p_{s_j} \) if \( p_{s_j} \leq \frac{1}{1 + \delta} v_i \).

**Uniqueness:** As before, we refer the situation where the manufacturers (suppliers) propose prices first as \( m \)-game (\( s \)-game) and let \( S_{m_i}, I_{m_i} \) the supremum and infimum of subgame perfect Nash equilibrium for manufacturers in a \( m \)-game (similarly, \( S_{s_j}, I_{s_j} \) for suppliers in a \( s \)-game), when all four firms are still in the market. By existence part, we already know that \( S_{m_2}, S_{s_2} \geq \frac{1}{1 + \delta} v_2 \geq I_{m_2}, I_{s_2} \) and \( S_{m_1}, S_{s_1} \geq \frac{1}{1 + \delta} v_1 \geq I_{m_1}, I_{s_1} \).

By using an approach similar to the previous part, we can show that the following inequalities have to hold in the equilibrium.

\[
\begin{align*}
I_{m_2} & \geq v_2 - \delta \max \left\{ \frac{1}{1 + \delta} v_2, S_{s_2} \right\} \quad (5) \\
S_{s_2} & \leq v_2 - \delta \min \left\{ \frac{1}{1 + \delta} v_2, I_{m_2} \right\} \quad (6) \\
I_{m_1} & \geq v_1 - \delta \max \left\{ \frac{1}{1 + \delta} v_1, S_{s_1} \right\} \quad (7) \\
S_{s_1} & \leq v_1 - \delta \min \left\{ \frac{1}{1 + \delta} v_1, I_{m_1} \right\} \quad (8)
\end{align*}
\]

Once we apply the existence conditions, this inequalities reduce to

\[
\begin{align*}
I_{m_2} & \geq v_2 - \delta S_{s_2} \\
S_{s_2} & \leq v_2 - \delta I_{m_2} \\
I_{m_1} & \geq v_1 - \delta S_{s_1} \\
S_{s_1} & \leq v_1 - \delta I_{m_1}
\end{align*}
\]

These conditions together with the existence conditions imply the desired equalities. We omit the proofs since they are similar to the one in the previous part.

**Proof of Proposition 3.**

We prove the result by induction. First, suppose that \( |M| \leq t, |S| \leq t \) with \( t = 2 \). We have already shown in Section 3 that the result is true for \( t = 2 \). Now, suppose that the result is true for all graphs with at most \( t = n - 1 \) firms on one side of the supply chain. That is, as the induction hypothesis, we assume that the result is true for graphs of size \( |M| \leq n - 1 \), \( |S| \leq n - 1 \). We are going to show that the result is true for any graph \( G \) of size \( |M| = |S| = n \).

We first start with the existence of the equilibrium. The following strategies construct a subgame perfect Nash equilibrium.

**Existence:** If we are in a strict subgraph of \( G \), that means at least one pair of firms has traded and left the supply chain network. Thus, the number of firms is strictly smaller than \( n \) both in \( M \) and \( S \). By the induction
hypothesis, we know that there exists a subgame perfect Nash equilibrium in this subgame, which is unique in terms of payoffs. The equilibrium strategies will follow the ones in any such subgames.

If we are in \( G \), that means no one has traded. Apply the algorithm and identify the subgraphs of types \( G^S \), \( G^M_s \), and \( G^E_k \). As before, call the subgames in which manufacturers are the proposers as \( m \)-game and the ones in which suppliers are the proposers as \( s \)-game.

For future reference, let \( P \) denote the set of price proposal such that

- In \( G^S \) type subgraphs, all proposed prices are 0 (in a \( m \)-game or in a \( s \)-game).
- In \( G^E_k \) type subgraphs, all proposed prices are \( \frac{\delta}{1+\delta} v_k \) in a \( m \)-game and \( \frac{1}{1+\delta} v_k \) in a \( s \)-game.
- In \( G^M_s \) type subgraphs, all proposed prices are \( v_s \) (in a \( m \)-game or in a \( s \)-game).

If the price proposal is equal to \( P \), then all responders accept (both in \( m \)-game and \( s \)-game). Notice that suppliers in \( s \)-game have incentives to ask higher prices only, while manufacturers in \( m \)-game would like to reduce the prices if they ever decide to deviate. Now, suppose that only one proposer deviates from the price proposal \( P \).

- If it is a \( s \)-game and the deviating supplier, say \( s_j \), belongs to a \( G^S \) type of subgraph, then all manufacturers in that subgraph accept 0.
- If it is a \( m \)-game and the deviating manufacturer, say \( m_i \), belongs to a subgraph of type \( G^M_s \), then all suppliers in that subgraph accept \( \tilde{v}_s \).
- If it is a \( s \)-game and the deviating supplier, say \( s_j \), belongs to a subgraph of type \( G^E_k \), then all neighbors of \( s_j \) reject the proposal of \( s_j \) and accept \( \frac{1}{1+\delta} v_k \) (if they can, otherwise they reject all offers) while all the other manufacturers hold onto their earlier decisions.
- If it is a \( m \)-game and the deviating manufacturer, say \( m_i \), belongs to a subgraph of type \( G^E_k \), then all neighbors of \( m_i \) reject the proposal of \( m_i \) and accept \( \frac{\delta}{1+\delta} v_k \) (if they can, otherwise they reject all offers) while all the other suppliers hold onto their earlier decisions.

Until now, we have only specified what proposers should propose, what responders should do when they face the price proposal \( P \), and how should responders react when they face some of the possible unilateral deviations. The remaining duty is to determine the reactions of responders in all the other cases. We define strategies so that if not all the possible number of pairs forms, then they follow the subgame perfect Nash equilibrium of the resulting subgraph (which we know exists by the induction step). If all firms reject, then the strategies will prescribe for proposers to propose price distribution \( P \) and for responders to accept. Therefore, we can conclude that given an action for all responders, the payoffs are immediately determined. For a given distribution of prices, the game is a one-shot game.
with a finite set of actions. This game must have at least one Nash equilibrium (possibly, in mixed strategies). We will define the strategies as follows: for a given distribution of prices, strategies will tell responders to play according to this Nash equilibrium. Notice that we may have a multiplicity of Nash equilibria. If this is the case, strategies must specify which of the several Nash equilibria will be played. Any specification will do the job.

Next, we consider the uniqueness of the equilibrium.

**Uniqueness:** — If $\tilde{G}$ is a $G^S$ type of supgraph, then the suppliers in $\tilde{G}$ (i.e., $\tilde{S}$) are collectively linked in $G$ only to the manufacturers in $\tilde{G}$ (i.e., $\tilde{M}$). We can separate the suppliers in $\tilde{G}$ into two sets: Set $\tilde{S}^+$, the suppliers who get a strictly positive payoff in the equilibrium, and set $\tilde{S}^0$, the suppliers who get zero payoff. It is straightforward to see that $\tilde{S}^+ \cup \tilde{S}^0 = \tilde{S}$ and $|\tilde{S}^0| \geq |\tilde{S}| - |\tilde{M}|$ since a positive payoff can only happen through trading for a positive price.

On the contrary, suppose that $|\tilde{S}^0| < |\tilde{S}|$ (i.e., not all suppliers in $\tilde{G}$ gets a zero payoff in the equilibrium). Then, there exists a supplier belonging to $\tilde{S}^0$ (call him $s^0$) that is linked to a manufacturer $m_i$ who pays a positive price in the equilibrium. To see why, first note that suppliers in $\tilde{S}^+$ must get a strictly positive payoff. Let $\tilde{M}^+$ be the set of manufacturers that trade with $\tilde{S}^+$ in the equilibrium. Because $\tilde{G}$ is of type $G^S$, $N_{\tilde{G}}(\tilde{M}^+) > |\tilde{M}^+|$ must hold by definition. Thus, there must be a supplier in $\tilde{S}^0$ who is linked to a manufacturer in $\tilde{M}^+$.

We now show that supplier $s^0$ has a profitable deviation, i.e., $s^0$ proposes a price of $\varepsilon > 0$ and some manufacturer accepts the price. Let $P^0$ be the distribution of prices that coincides with $P$ except for the price proposed by $s^0$, and suppose that the deviation is not profitable. This implies that when facing distribution of prices $P^0$, the manufacturer $m_i$ does not accept the price $\varepsilon$. Then, it must be the case that $m_i$ is able to accept and trade for price zero with a supplier, say $s^1$ in $N_{\tilde{G}}(m_i)$.

Say that there are $n$ suppliers (where $1 \leq n \leq |S|$) in $G$ that propose zero in the distribution $P$ ($s^1$ is one of them). We know that when facing $P$ manufacturer $m_i$ cannot trade for price zero. Then, it must be the case that another manufacturer $m_j$ with $v_j > v_i$ accepts zero and trades with $s^1$. Moreover, when facing $P^0$, it must be the case that $m_j$ can trade for price zero with another supplier, say $s^2$, since $m_i$ is able to trade for price zero in such a case.

But now we can repeat the reasoning above and state that when facing $P$, it must be the case that there exists another manufacturer $m_k$ with $v_k > v_j$ who accepts zero and trades with $s^2$ (if not, both $m_i$ and $m_j$ could accept zero and trade for price zero). This implies in turn that when facing $P^0$ it will be the case that $m_k$ can trade for price zero with another supplier, say $s^3$. We can iterate this procedure until we conclude that there must exist $n + 1$ different suppliers that propose zero, which is a contradiction since there are $n$ suppliers proposing zero in $P$. 

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If $\tilde{G}$ is a $G_k^E$ type of supgraph, then, by definition, the lowest valuation in $\tilde{G}$ is $\tilde{v}_k$. Suppose that, in an equilibrium, a manufacturer and supplier trade at a price higher than $\tilde{v}_k$. Let $p$ be the price and suppose that supplier $s$ and manufacturer $m_i$ with valuation $v_i (> \tilde{v}_k)$ trade at $p$. Notice that, because $m_i$ agrees to pay $p$ in the equilibrium, it must be the case that he cannot lever the valuation difference between $v_i$ and $\tilde{v}_k$ because either $m_i$ has either only one link (i.e., cannot lever due to link restriction) or the lowest valuation among all her competitors (i.e., cannot lever due to valuation restriction). Thus, by definition, $m_i$ is a moderate manufacturer that has a valuation higher than $\tilde{v}_k$ (since she trades at $p > \tilde{v}_k$). But, this is a contradiction because $\tilde{G}$ is a $G_k^E$ type of subgraph and, by definition, the only moderate manufacturer in a $G_k^E$ subgraph is the manufacturer with the valuation $\tilde{v}_k$.

Finally, suppose that $\tilde{G}$ be a $G_s^M$ type of supgraph. We first show that, in an equilibrium, all the suppliers in $\tilde{G}$ receive the same payoff. Notice that every supplier in $\tilde{G}$ trades with some manufacturer in every equilibrium since there are more manufacturers than suppliers in $\tilde{G}$ by definition. Now, suppose that in an equilibrium, there exists $k$ different prices. Let $p$ be the highest equilibrium price and suppose that supplier $s_j$ and manufacturer $m_i$ trade at $p$. First, consider the critical soft manufacturer $m_s$ with the valuation $\tilde{v}_s$. Any manufacturer in $C_{\tilde{G}}(m_s)$ must have proposed at least a price of $v_s$. Otherwise, $m_s$ would engage in trade by offering $v_s$. Because $p$ is the highest price in the equilibrium, we must have $v_i - p < v_i - v_s$, or equivalently $p > v_s$. On the other hand, because $p$ is an equilibrium price, it must be the case that $v_i - p \geq \delta(\frac{1}{1+\delta}v_i)$ where the last part of the inequality is the value that $m_i$ can obtain when all suppliers reject his offer. By reorganizing the inequality, we get $p \leq \frac{1}{1+\delta}v_i$. Recall that $v_s \geq \frac{1}{1+\delta}v_i$ for any $v_i$ by assumption. So, we must have $p \leq \tilde{v}_s$, which is a contradiction. Thus, in any equilibrium, the suppliers offer the same price. Then, clearly, the equilibrium in which all suppliers offer $\tilde{v}_s$ is Pareto dominant for the suppliers.

**Proof of Proposition 4.** First, suppose that the supply chain network $(G, v)$ decomposes into a unique type of subnetwork. If $(G, v)$ decomposes into $(G, v)^S$ type of subnetworks only, all the transactions will be priced at zero by Proposition 3. Furthermore, the only way that $(G, v)$ decomposes into a $(G, v)^E_k$ or $(G, v)^M_s$ type of subnetwork is that the decomposition returns $(G, v)$ itself. Then, by Proposition 3, it is easy to see that the supply chain network $(G, v)$ has to be competitive (notice that $v^* = v_s$ is the $(|S| + 1)$th highest valuation in the network when the network is of type $(G, v)^M_s$).

Now, we show that if $(G, v)$ is competitive then $(G, v)$ decomposes into a unique type of subnetworks. On the contrary, suppose that $(G, v)$ is competitive but it decomposes into subnetworks with at least two different types when we apply the decomposition algorithm. Then, by Proposition 3, there must be at least two different prices in the equilibrium (notice that $G_k^E$ and $G_l^E$ or $G_s^M$ and $G_r^M$ are different types of
subnetworks), which is a contradiction. Thus, if \((G, v)\) is competitive it must decompose into a unique type of subnetwork.

**Proof of Proposition 6.** Suppose that the manufacturer \(m_i\) and supplier \(s_j\) belongs to the graph \(G\), which does not contain the link \(ij\).

(i) (1) Suppose that after the decomposition algorithm is applied to \(G\), manufacturer \(m_i\) and supplier \(s_j\) belongs to the same subgraph \(G^*\), which does not contain the link \(ij\). This subgraph is non-deficient since otherwise, the algorithm would decompose it further into smaller subgraphs. Then, \(G^* + ij\) would still be non-deficient since adding a link would only help to be non-deficient. Therefore, applying the decomposition to \(G + ij\) would give the same result as applying the algorithm to \(G\). Thus, by Proposition 3, we must have \(\rho(G + ij) = \rho(G)\).

(2) Suppose that after the decomposition algorithm is applied to \(G\), manufacturer \(m_i\) belongs to \(G^S\) type of subgraph, say \(G_i\). By part (i)-(1), we know that if \(m_i\) and \(s_j\) belongs to the same subgraph we have \(\rho(G + ij) = \rho(G)\). Thus, assume that \(m_i\) and \(s_j\) belongs different type of subgraphs \(G_i\) and \(G_j\), respectively. Notice that, if \(G_j\) is of type \(G^E_k\), then \(G_i\) would still be a \(G^S\) type of subgraph when the algorithm is applied to \(G + ij\). Thus, the number of different prices does not change when we add the link \(ij\) into \(G\) and \(\rho(G + ij) = \rho(G)\). Now, assume that \(G_j\) is a \(G^M_s\) type of subgraph. Then, notice that the decomposition algorithm would still identify \(G_j\) as a \(G^M_j\) type when the algorithm is applied to \(G + ij\). So, the price outcome would be the same as in network \(G\).

(3) Suppose that after the decomposition algorithm is applied to \(G\), supplier \(s_j\) belongs to \(G^M_s\) type of subgraph, say \(G_j\). By part (i)-(1), we know that if \(m_i\) and \(s_j\) belongs to the same subgraph we have \(\rho(G + ij) = \rho(G)\). Thus, assume that \(m_i\) and \(s_j\) belongs different type of subgraphs \(G_i\) and \(G_j\), respectively. Notice that, if \(G_i\) is of type \(G^E_k\), then \(G_j\) would still be a \(G^M_s\) type of subgraph when the algorithm is applied to \(G + ij\). Thus, the number of different prices does not change when we add the link \(ij\) into \(G\) and \(\rho(G + ij) = \rho(G)\). Now, assume that \(G_i\) is a \(G^S\) type of subgraph. In this case, by part (i)-(2), we know that the price outcome would be the same as in network \(G\).

(ii) (1) Suppose that after the decomposition algorithm is applied to \(G\), the manufacturer \(m_i\) belongs to the \(G^E_k\) type of subgraph \(G_i\) and the supplier \(s_j\) belongs to the \(G^E_l\) type of subgraph \(G_j\). Notice that when applied to \(G + ij\), the algorithm will identify the union of \(G_i\) and \(G_j\) graphs as a \(G^E\) type of subgraph. Whether the algorithm will decompose this \(G^E\) type subgraph into further into smaller subgraphs depends on the number of moderate manufacturers.
(a) Suppose that \( m_i = m_k, m_l \in N_{G+ij}(s_j) \), and \( |N_{G+ij}(m_l)| > 1 \). First, since both \( m_k \) and \( m_l \) are neighbors of \( s_j \) so that \( m_k \) and \( m_l \) are direct competitors. If \( v_k > v_l \), then \( v_k \) cannot be a moderate manufacturer anymore (since \( m_k \) has more than one link together with \( ij \) and not have the lowest valuation among all her competitors). On the other hand, if \( v_k < v_l \), then \( m_l \) cannot be a moderate manufacturer. Thus, when both \( m_k \) and \( m_l \) are neighbors of \( s_j \), we must have \( \rho(G + ij) < \rho(G) \).

(b) Suppose that \( m_i = m_k, m_l \in N_{G+ij}(s_j) \), \( |N_{G+ij}(m_l)| = 1 \), and \( v_k > v_l \). Since both \( m_k \) and \( m_l \) are neighbors of \( s_j \) so that \( m_k \) and \( m_l \) are direct competitors. Because \( m_l \) has only one link, he is still a moderate manufacturer. Because \( v_k > v_l \), the manufacturer \( m_k \) cannot be a moderate manufacturer anymore (since \( m_k \) has more than one link together with \( ij \) and not have the lowest valuation among all her competitors). Thus, \( \rho(G + ij) < \rho(G) \).

(c) Suppose that \( m_i = m_k, m_l \notin N_{G+ij}(s_j) \), and \( m_k \) does not have the lowest valuation in \( N_{G+ij}(s_j) \). Because \( m_l \) is not a neighbor of \( s_j \), \( m_l \) is still a moderate manufacturer in \( G + ij \). Because \( m_k \) does not have the lowest valuation in \( N_{G+ij}(s_j) \), \( m_k \) is not a moderate manufacturer. Thus, we must have \( \rho(G + ij) < \rho(G) \).

(d) Suppose that \( m_i \neq m_k, m_l \in N_{G+ij}(s_j) \), \( |N_{G+ij}(m_l)| > 1 \), and \( m_l \) does not have the lowest valuation in \( N_{G+ij}(s_j) \). Because \( m_k \) is not a neighbor of \( s_j \), \( m_k \) is still a moderate manufacturer in \( G + ij \). Because \( m_l \) does not have the lowest valuation in \( N_{G+ij}(s_j) \), \( m_l \) is not a moderate manufacturer. Thus, we must have \( \rho(G + ij) < \rho(G) \).

(2) Suppose that after the decomposition algorithm is applied to \( G \), manufacturer \( m_i \) belongs to the \( G^M_s \) type of subgraph \( G_i \), which has only one soft manufacturer. Also, suppose that \( s_j \) belongs to the subgraph \( G_j \).

(a) Suppose that \( G_j \) is of type \( G^E_k \). Notice that if we remove \( ij \) from \( G_i \), then \( G_i - ij \) can only be \( G^E \) type. Then, the decomposition algorithm will identify the graph \( G_i + G_j + ij \) as a \( G^M_x \) type where \( x \) represents the relevant moderate manufacturer. Thus, \( \rho(G + ij) < \rho(G) \) in this case.

(b) Assume that \( G_j \) is of type \( G^S \) and \( |S_i + S_j| = |M_i + M_j| \). Notice that if we remove \( ij \) from \( G_i \), then \( G_i - ij \) can only be \( G^E \) type. Also, if we remove \( ij \) from \( G_j \), then \( G_j - ij \) can be either \( G^S \) or \( G^E \) type. Notice that when \( G_j + ij \) has equal number of suppliers and manufacturers, the algorithm will not identify any \( G^S \) or \( G^M \) type of subgraphs and \( G_i + G_j + ij \) will be a \( G^E \) type, in which case \( \rho(G + ij) < \rho(G) \).

**Proof of Proposition 7.** First suppose that \( (G,v) \) is efficient. If \( (G,v) \) is of type \( (G,v)^E_k \) or \( (G,v)^S \) then all manufacturers receive a good and there are no soft manufacturers in \( (G,v) \) by definition. So, suppose
that \((G, v)\) is of type \((G, v)_S^M\). Because \((G, v)\) is efficient, the manufacturers with the highest \(|S|\) valuations receive the good in the equilibrium of the bargaining game. Thus, all the manufacturers who have valuations less than or equal to the \((|S| + 1)\)th highest manufacturer valuation is a soft manufacturer.

Now, suppose that after decomposition, either there are no soft manufacturers or all soft manufacturers have valuations less than or equal to the \((|S| + 1)\)th highest manufacturer valuation. If there are no soft manufacturers, then \((G, v)\) must be either \((G, v)_E^k\) or \((G, v)_S^S\) type. In either case, all the manufacturers receive a good in the equilibrium and \((G, v)\) is efficient. So, suppose that there are soft manufacturers in \((G, v)\) and all soft manufacturers have valuations less than or equal to the \((|S| + 1)\)th highest manufacturer valuation. Then, only the manufacturers with the highest \(|S|\) valuations receive the good in the equilibrium of the bargaining game, which implies that \((G, v)\) is efficient.

**Proof of Proposition 8.** Suppose that \((G, v)\) supports the competitive wholesale prices but it is not efficient. Because \((G, v)\) is competitive, by Proposition 4, we know that \((G, v)\) must decompose into a unique type of subnetworks. If \((G, v)\) is of type \((G, v)_E^k\) or decomposes into subnetworks of type \((G, v)_S^S\) then all manufacturers receive a good, which is a contradiction since \((G, v)\) is assumed not to be efficient. Thus, suppose that \((G, v)\) is of type \((G, v)_S^M\), where \(m_s\) is the critical soft manufacturer. Because \((G, v)\) is not efficient, there must a manufacturer \(m_k\) such that \(v_k > v_s\) and \(m_k\) cannot receive a good in the equilibrium. The only way \(m_k\) cannot engage in trade is that all the competitors of \(m_k\) must have higher valuations than \(v_k\); i.e., \(m_k\) has the lowest valuation among her competitors. Then, the suppliers in \(N_G(m_k)\) will sell their goods for the price \(v_k\). This is a contradiction since, by Proposition 4, we know that \(v_s\) must be the only price in the equilibrium.

**Proof of Proposition 9.** Suppose that the manufacturer \(m_i\) and supplier \(s_j\) belongs to the graph \(G\), which does not contain the link \(ij\).

(i) (1) Suppose that after the decomposition algorithm is applied to \(G\), manufacturer \(m_i\) and supplier \(s_j\) belongs to the same subgraph \(G^*\), which does not contain the link \(ij\). This subgraph is competitive and, by Proposition 8, it is also efficient. Then, \(G^* + ij\) would still be competitive and efficient since adding a link would only help to improve competitiveness. Therefore, applying the decomposition to \(G + ij\) would give the same result as applying the algorithm to \(G\) and \(\mu(G + ij) = \mu(G)\).

(2) Suppose that both \(m_i\) and \(s_j\) belongs to a subnetwork of type \(G^S\) or \(G^E_k\). Notice that, in this case, applying the algorithm to \(G + ij\), produces an outcome where each subnetwork is either of type \(G^S\) or \(G^E_k\) (i.e., combining \(G^S\) and \(G^E_k\) type of graphs cannot convert the graphs into a \(G^M\) type). Thus, in both \(G\) and \(G + ij\), all manufacturers receive a good in the equilibrium and \(\mu(G + ij) = \mu(G)\).
(3) Suppose that after the decomposition algorithm is applied to $G$, supplier $s_j$ belongs to $G^M_s$ type of subgraph. By part (i)-(1), we know that if $m_i$ and $s_j$ belongs to the same subgraph we have $\mu(G + ij) = \mu(G)$. Thus, assume that $m_i$ and $s_j$ belongs different type of subgraphs $G_i$ and $G_j$, respectively. Notice that, if $G_i$ is of type $G^E_k$ or $G^S$, then $G_j$ would still be a $G^M_s$ type of subgraph when the algorithm is applied to $G + ij$. Thus, we must have $\mu(G + ij) = \mu(G)$ since all the transactions will be the same as in $G$.

(ii) (1) Suppose that after the decomposition algorithm is applied to $G$, manufacturer $m_i$ belongs $G_i$, which is of type $G^M_s$ with one soft manufacturer, and $s_j$ belongs to the subgraph $G_j$, which is of type $G^E_k$. Notice that if we remove $ij$ from $G_i$, then $G_i - ij$ can only be $G^E$ type. Then, the decomposition algorithm will identify the graph $G_i + G_j + ij$ as a $G^M_s$ type where $x$ represents the relevant moderate manufacturer. Notice that $x$ must be such that $v_x < v_s$, since in $G_i + G_j + ij = G_{ij}$, it must be the case that $m_s$ (which is the critical soft manufacturer in $G_i$) will receive a good and $m_k$ (which is the moderate buyer in $G_j$) will not. Therefore, we must have $\mu(G) < \mu(G + ij)$ since $v_s > v_k$.

(2) Suppose that after the decomposition algorithm is applied to $G$, manufacturer $m_i$ belongs $G_i$, which is of type $G^M_s$ with one soft manufacturer, $s_j$ belongs to the subgraph $G_j$, which is of type $G^E_k$, and $|S_i + S_j| = |M_i + M_j|$. Notice that if we remove $ij$ from $G_i$, then $G_i - ij$ can only be $G^E$ type. Also, if we remove $ij$ from $G_j$, then $G_j - ij$ can be either $G^S$ or $G^E$ type. Notice that when $G_j + ij$ has equal number of suppliers and manufacturers, the algorithm will not identify any $G^S$ or $G^M$ type of subgraphs and $G_i + G_j + ij$ will be a $G^E$ type, in which case all the manufacturers will receive a good in the equilibrium. Since an additional supplier and manufacturer trades in $G_{ij}$, we must have $\mu(G) < \mu(G_{ij})$.

(3) Suppose that after the decomposition algorithm is applied to $G$, manufacturer $m_i$ belongs $G_i$, which is of type $G^M_s$, and $s_j$ belongs to the subgraph $G_j$, which is of type $G^E_k$. Notice that if we remove $ij$ from $G_i$, then $G_i - ij$ can only be $G^M_s$ type of subgraph. Then, the algorithm will identify $G_i - ij$ and $G_j + ij$ as $G^M_s$ and $G^M_k$, respectively. The reason that $G_j + ij$ will be a $G^M_k$ type is because after $G_j + ij$ is non-deficient, the number manufacturers in $G_j + ij$ is exactly one more than the number of suppliers and $m_k$ has the lowest valuation in $G_j + ij$ (since $v_j > v_k$). Therefore, we must have $\mu(G) < \mu(G_{ij})$.

B. Decomposition Algorithm

→ Part 1.

- Step $m_1$.  

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– Step $m_1$.

* step $m_1.1$. We start from $G_t = (M_t, S_t, L_t)$ with the initial graph being $G_1 = G$ with $|S|$ suppliers and $|M|$ manufacturers. Look at every subset $M'$ of $M_t$ such that $|M'| = 2$, starting from the subsets that contain $m_1$ in the order $\{m_1, m_2\}, \{m_1, m_3\}, \ldots, \{m_1, m_{|M'|}\}$, then with the ones that contain $m_2$ in the order $\{m_2, m_3\}, \{m_2, m_4\}, \ldots, \{m_2, m_{|M'|}\}$ and so on. That is, the order for looking at the subsets is $M$ once you find one $M' \subseteq M_t$ with $|M'| = 2$ such that $|N(M')| = 1$, stop. For every manufacturer $m_i \notin M'$ (again here we will follow the ordering given by their subindexes), if it is true that $N(M' \cup m_i) = N(M')$, then relabel $M' := M' \cup m_i$.

Call $G^1_t$, (superscript 1 stands for “Part 1”) the subgraph in $G_t$ induced by the set of manufacturers $M'$ and the set of suppliers $N(M')$.

* step $m_1.2$. If we run step $m_1.1$ and we found a $G^1_t$, then call $\bigcup_{j=t+1}^{k_t} G_j := G_t - G^1_t$, i.e., the connected subgraphs that we get when we remove $G^1_t$ from $G_t$, and run again step $m_1$ with each $G_j$ with $j > t$.

If we run step $m_1.1$ without finding any $G^1_t$, then go to step $m_1.2$. ...

– Step $m_1$.k.

* step $m_1.k.1$. We start from $G_t$. Look at every subset $M'$ of $M_t$ such that $|M'| = k + 1$, following the lexicographic ordering.

Once you find one $M' \subseteq M_t$ with $|M'| = k + 1$ such that $|N(M')| = k$, stop. For every manufacturer $m_i \notin M'$ (again here we will follow the ordering given by their subindexes), if it is true that $N(M' \cup m_i) = N(M')$, then relabel $M' := M' \cup m_i$.

Call $G^1_t$, the subgraph in $G_t$ induced by the set of manufacturers $M'$ and the set of suppliers $N(M')$.

* step $m_1.k.2$. If we run step $m_1.k.1$ and we found a $G^1_t$, then call $\bigcup_{j=t+1}^{k_t} G_j := G_t - G^1_t$, i.e., the connected subgraphs that we get when we remove $G^1_t$ from $G_t$, and go again to step $m_1$ with each $G_j$ with $j > t$.

If we run step $m_1.k.1$ without finding any $G^1_t$, then go to step $m_1.k + 1$.

...

– Step $m_1|M|$.

* step $m_1|M|.1$. We start from $G_t$. Look at every subset $M'$ of $M_t$ such that $|M'| = |S| + 1$, following the same ordering as before.

Once you find one $M' \subseteq M_t$ with $|M'| = |S| + 1$ such that $|N(M')| = |S|$, stop. For every manufacturer $m_i \notin M'$ (again here we will follow the ordering given by their subindexes), if
it is true that \( N(M' \cup m_i) = N(M') \), then relabel \( M' := M' \cup m_i \).

Call \( G^1_t \), the subgraph in \( G_t \) induced by the set of manufacturers \( M' \) and the set of suppliers \( N(M') \).

* step \( m_1 \mid M \). If we run step \( m_1 \mid M \).1 and we found a \( G^1_t \), then call \( \bigcup_{j=t+1}^{k_1} G_j := G_t - G^1_t \), i.e., the connected subgraphs that we get when we remove \( G^1_t \) from \( G_t \), and go again to step \( m_1 \).1 with each \( G_j \) with \( j > t \).

If we run step \( m_1 \mid M \).1 without finding any \( G^1_t \), then label \( D^1_t = \bigcup_{j=t+1}^{k_1} G_j := G_t - G^1_t \), i.e., the decomposition that we get when we start the algorithm with \( m_1 \). Find the surplus maximizing matching (which is a maximum weighted matching where the weights are the manufacturer valuations and can be found by the famous Hungarian algorithm provided by Kuhn, 1955) and label it as \( v(D^1_t) \), i.e., the surplus created by the decomposition \( D^1_t \).

- Step \( m_2 \). Repeat the process in Step \( m_1 \) but this time start with manufacturer \( m_2 \). In particular, look at every subset \( M' \) of \( M_t \) such that \( |M'| = 2 \), starting from the subsets that contain \( m_2 \) in the order \( \{m_2, m_3\}, \{m_2, m_4\}, ..., \{m_2, m_{|M|}\}, \{m_2, m_1\} \), then with the ones that contain \( m_3 \) in the order \( \{m_3, m_4\}, \{m_3, m_5\}, ..., \{m_3, m_{|M|}\}, \{m_3, m_1\} \) and so on. The rest of the process is identical to the one in Step \( m_1 \).

- ...

- Step \( m_{|M|} \mid M \). Repeat the process in Step \( m_1 \) but this time start with manufacturer \( m_{|M|} \). Find the decomposition that gives the highest total surplus. That is, find \( D = \arg \max_{D_j \in \{D_1, ..., D_{m_1}\}} v(D_j) \). Label \( G_t = G_t - D \) and go to Part 2.

→ Part 2.

Part 2 is completely symmetric to Part 1, with the roles of manufacturers and suppliers get reversed. The steps go from step \( s_1 \) to step \( s_{|S|} \). We start Part 2 with the \( G_t \) that comes from the last iteration in Part 1. The subgraphs that we remove will now be called \( G^2_t \), (superscript 2 is for Part 2).

→ Part 3.

We start Part 3 with the \( G_t \) that comes from the last iteration in Part 2, which is the set of subgraphs that are not removed from \( G \) in Part 1 or Part 2. Find moderate manufacturers as defined in Definition 2. Relabel the moderate manufacturers as \( mm_1, ..., mm_k \) where \( mm_1 \) has the highest valuation and \( mm_k \) has the lowest. Remove links between \( N(mm_1) \) and \( \{m_i \in N(N(mm_1)) : v_{m_i} < v_{mm_1}\} \). Label the component which contains \( mm_1 \) as \( G^E_k \) where \( k \) is the index of the manufacturer \( mm_1 \) in the original network. Continue
removing links and labeling components in this fashion for each moderate manufacturers. If there is no moderate manufacturers left, stop.

→ End of the algorithm.